CS698A: Selective Areas of Mechanism Design	**Jan-Apr 2018**

Lecture 13: Two-sided Matching

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13.1 Recap and Introduction

In the last few lectures, we looked at the one-sided matching problem and presented the popular Top Trading Cycle Mechanism which satisfies certain desirable properties. Now, we will consider the problem where both the sets (that are to be matched) will have preference ordering over each other. This two-sided matching problem is more widely used due to its extensive application domain.

13.1.1 Examples of Two-sided Matchings

- Marriage and Dating markets
- Allocation of Students to Medical Residencies
- University-Student matching (IITJEE seat allocation)
- Job market: Employers and Candidates

We will begin by considering the case where each entity from either of the two sets is to be matched to exactly one entity from the other set. We will refer to this as the *Marriage problem.*

13.2 Marriage problem: Setup and Notations

Let M denote the set of men, W denote the set of women. For simplicity, let us assume that $|M| = |W|$. However, this is not a necessary assumption and we can extend the results for general case. Let P_m denote the strict preference order over W for a player $m \in M$. Similarly, P_w be the corresponding strict preference order over M for $w \in W$. Formally, P_m can be defined as follows:

$$xP_my \Leftrightarrow m \text{ strictly prefers } x \text{ over } y,\ x, y \in W$$

Definition 13.1 (Matching) *A matching is a bijection $\mu : M \to N$. Accordingly, $\mu(m) \in W$ is the woman assigned to $m \in M$ and $\mu^{-1}(w) \in M$ is the man assigned to $w \in W$.*

Example A: Consider $M = \{m_1, m_2, m_3\}$, $W = \{w_1, w_2, w_3\}$. Let the preference orderings be as tabulated in Table 13.1. A sample candidate matching be given by $\mu : \mu(m_i) = w_i$ for $i = 1, 2, 3$. Notice that

$$w_2 P_{m_3} \mu(m_3) \quad \text{and} \quad m_3 P_{w_2} \mu^{-1}(w_2)$$

P_{m_1}	P_{m_2}	P_{m_3}	P_{w_1}	P_{w_2}	P_{w_3}
w_2	w_1	w_1	m_1	m_3	m_1
w_1	w_3	w_2	m_3	m_1	m_3
w_3	w_2	w_3	m_2	m_2	m_2

Table 13.1: Preference profile

Thus, (m_3, w_2) can move out and *block* this matching. This leads us to define the first desirable property of two-sided matching, *Stability.*

Definition 13.2 (Stable Matching) *A matching μ is said to be pairwise unstable at a preference profile P if $\exists\ m, m^{'} \in M$ s.t.*

$$\text{(i) } \mu(m')P_m\mu(m) \ \text{ (ii) } mP_{\mu(m')}m'$$

The pair $(m, \mu(m'))$ is called a blocking pair of μ at P. If a matching has no blocking pairs at a preference profile P, then it is said to be a stable matching at P.

Example: (Stable Candidate match) For the previous example, consider the matching

$$m_1 \to w_1, m_2 \to w_3, m_3 \to w_2$$

Note that this is a stable matching for the given preference profile. At this point, the following questions may be natural to ask:

- Will there always exist a pairwise stable matching for a given preference profile?
- Why are we concerned with only *pairwise stability* and not *coalitional stability* as done in one-sided matching?

This leads us to the notion of *Group blocking* - a group may want to redistribute their initial endowments and be better off.

Definition 13.3 (Group Blocking) *A coalition $S \subseteq M \cup W$ is said to be a blocking a matching μ at a preference profile P if $\exists$ another matching $\mu^{'}$ s.t.*

$$\text{(i) } \forall m \in M \cap S,\ \mu^{'}(m) \in W \cap S \quad \text{and} \quad \forall w \in W \cap S,\ \mu^{'-1}(w) \in M \cap S$$

$$\text{(ii) } \forall m \in M \cap S, \mu^{'}(m)P_m\mu(m) \quad \text{and} \quad \forall w \in W \cap S, \mu^{'-1}(w)P_w\mu^{-1}(w)$$

Definition 13.4 (Matching in the Core) *A matching μ is in the core of the induced coalitional game at a profile P if no coalition can block μ at P.*

The following result shows that the condition for group blocking is equivalent to pairwise blocking.

Theorem 13.5 *A matching is pairwise stable iff it belongs to the core of the induced coalitional game at the given profile.*

Proof: $(\Leftarrow)$ *(Trivial) If no coalition of any size can block the matching, in particular, no pair can block the matching. Thus, the matching will be pairwise stable.*

($\Rightarrow$) *Let μ be a pairwise stable matching. Assume, for contradiction that μ is not in the core. Thus, $\exists$ $S \subseteq M \cup W$ and a matching $\hat{\mu}$ s.t. $\forall m \in M \cap S$ and $\forall w \in W \cap S$ with $\hat{\mu}(m), \hat{\mu}^{-1}(w) \in S$, we have $\hat{\mu}(m) P_m \mu(m)$ and $\hat{\mu}^{-1}(w) P_w \mu^{-1}(w)$. This implies that $\exists m \in M \cap S$ s.t. $\hat{\mu}(m) \in W \cap S$. Let $\hat{\mu}(m) = w$, thus*

$$w P_m \mu(m) \quad and \quad m P_w \mu^{-1}(w)$$

Hence, (m, w) is a blocking pair of μ at P. This is a contradiction to μ being pairwise stable at P. ■

Thus, from here on, we can consider *Stability* to be referring to *Pairwise Stability*. Now, we will head towards answering the question of existence of a stable matching for a given preference profile.

13.3 Deferred Acceptance Algorithm (Gale-Shapley)

A stable matching always exists in a Marriage market. This can be proved via exhibiting an algorithm to find such a matching. The basic idea of the algorithm given by Gale and Shapley (1962) is that in each iteration, one side proposes the other side of the market and the proposed agent may accept or reject the proposal. This will continue until certain stopping criteria are met. The two versions of the algorithm are: *Men-proposing* and *Women-proposing*.

13.3.1 Men-proposing Deferred Acceptance Algorithm

- **Step 1:** Every man proposes to his top-ranked woman.
- **Step 2:** Every woman who has at least one proposal *tentatively* keeps the top man among the recieved proposals and rejects the rest.
- **Step 3:** Every man who was rejected in the previous round, proposes to the top ranked woman in his preference order that has not rejected him in any of the earlier rounds.
- **Step 4:** Every woman who gets at least one proposal, including the tentatively accepted proposal again tentatively keeps the top man and rejects the rest. Note that the woman getting a better proposal in this round will reject previously (tentatively) accepted proposal.

  The process is repeated from Step 3 and will continue till each woman gets at least one proposal. When each woman has at least one proposal, the current tentatively accepted proposal becomes the final match and thus we get a match.

Example:

1. **(Applying M-P DA Algorithm on Example A)** See Table 13.2. The final matching is
$$\mu : \mu(m_1) = w_2, \mu(m_2) = w_3, \mu(m_3) = w_1$$
2. **(Applying W-P DA Algorithm on Example A)** See Table 13.3. The final matching is
$$\mu : \mu(m_1) = w_1, \mu(m_2) = w_3, \mu(m_3) = w_2$$

This it is evident from the above example that the matchings obtained by M-P DA Algorithm and W-P DA Algorithm may be different.

Illutration: For a programmed illustration of the DA algorithm, the reader may take a look at the app: The Advisor-Student Match App.

Table 13.2: MP-DA Algorithm: Stepwise

Proposals	(Tentative) Matches
$m_1 \to w_2$	(m_1, w_2)
$m_2 \to w_1$	(m_3, w_1)
$m_3 \to w_1$	Rejected

Round 1

Proposals	(Tentative) Matches
$m_2 \to w_3$	(m_2, w_3)

Round 2

Table 13.3: WP-DA Algorithm: Stepwise

Proposals	(Tentative) Matches
$w_1 \to m_1$	(m_1, w_1)
$w_2 \to m_3$	(m_3, w_2)
$w_3 \to m_1$	Rejected

Round 1

Proposals	(Tentative) Matches
$w_3 \to m_3$	Rejected

Round 2

Proposals	(Tentative) Matches
$w_3 \to m_2$	(m_2, w_3)

Round 3

13.3.1.1 Remarks

- Since each woman is allowed to keep only one proposal at every stage of the algorithm, no woman gets more than one match.
- Similarly, if a man's proposal is tentatively accepted, he is not allowed to propose further whcih ensures that each woman is assigned to exactly one man.
- **Claim 13.6 (Algorithm termination)** *The algorithm terminates in finite number of steps.*
  Proof: *Since the set of woman a man proposes does not increase for all men and strictly decreases for at least one man, the claim follows.* ∎
- The previous remarks and claim 1.6 also show that the algorithm terminates in a valid matching.

13.3.2 Stability and Optimality of the DA Algorithm

Theorem 13.7 (Stability of DA Algorithm) *At every preference profile, the DA algorithm terminates at a stable matching for that profile.*
Proof: *We will prove the result for the Men-proposing DA algorithm. Exactly similar proof will work for*

Women-proposing DA algorithm.
Let μ be the matching obtained by the algorithm. Assume for contradiction, μ is not stable. Hence, $\exists m \in M$ and $w \in W$ s.t. (m, w) is a blocking pair. By assumption, $w \neq \mu(m)$ and $wP_m\mu(m)$. This during the course of the algorithm, m must have proposed w at some stage and must have been rejected before being matched to $\mu(m)$. But w rejected m since she got a better proposal. Therefore, $\mu^{-1}(w)P_w m$. This contradicts the fact that (m, w) is a blocking pair. ■

Also, it is worth noting that the matching we will obtain by the (M-P or W-P) DA algorithm will be a unique one.

Some natural Questions:

- Men-proposing DA and Women-proposing DA may yield different stable matchings as noted for Example A. Is there an incentive to prefer one over another?
- How should we define desirable criterion for selecting one stable matching if we have multiple stable matchings for a given preference profile?

13.4 Summary

To summarize the lecture, we began with some motivating examples of Two-sided Matching problems. We formally defined the setup for two-sided matching and introduced the Deferred Acceptance algorithm to find a *stable* matching. We then proved that for a given preference profile, the matching obtained by the DA algorithm is stable. We ended by stating some naturally occuring questions regarding comparison among multiple stable matchings.