CS698A: Selected Areas of Mechanism Design **Jan-Apr 2018**

Lecture 12: Top Trading Cycle Mechanisms

Lecturer: Swaprava Nath *Scribe(s): Massand Sagar Sunil*

12.1 Top trading cycle with fixed endowments

Top trading cycle with fixed endowments is a mechanism for one-sided matching (e.g. house allocation) with a fixed endowment of items at the beginning. It is 'strategy-proof' and also satisfies 'stability' and 'efficiency'.

Algorithm 1 Top trading cycle with fixed endowments

procedure TOP TRADING CYCLE
  $M \leftarrow$ Set of houses to be reallocated
  $N \leftarrow$ Set of players who will be relocated.
  $P \leftarrow$ Preference ordering for each of the players
  $a^0 : N \rightarrow M$ //Mapping from players to their initial endowments.
  $M^1 \leftarrow M$
  $N^1 \leftarrow N$
  $a : N \rightarrow M$ //Mapping from players to their reallocated values.
  $i \leftarrow 1$
While$(N^i != \phi)$:
  $E^i \leftarrow \{(s,t) | s \in N^i, t \in N^i, P_s(1, M^i) = a^0(t)\}$
  $G^i \leftarrow (N^i, E^i)$
  For every cycle $C = (q^1, q^2, ..., q^{P-1}, q^P)$ in G^i, $a(q^i) = a^0(q^{(i\%P)+1})$.
  $\widehat{N^i} \leftarrow$ {s$|s \in N^i$, s is present in cycle C }
  $\widehat{M^i} \leftarrow$ {t$|s \in N^i, a^0(s) = t$}
  $N^{i+1} \leftarrow N^i - \widehat{N^i}$
  $M^{i+1} \leftarrow M^i - \widehat{M^i}$
  $i \leftarrow i + 1$

Theorem 12.1 *Top Trading Cycle (TTC) is strategy-proof and efficient.*

Proof: *(Strategy-proof)* Suppose, agent i is truthful, and gets a house in round k. The TTC allocation guarantees that agent i gets its highest preference from houses in M^k. Suppose agent i deviates from his true preference ordering. If this change results in an allocation for i in a round $r >= k$, the allocation due to the deviation is not strictly better than the allocation on being truthful. Thus, to complete the proof, we just need to prove that, if i is allocated a house in a round $r < k$ (due to deviation), the allocation isn't better. Let us assume that a house is allocated to player i in a round $r < k$. Consider $\Pi_i = \{j|$ There is a directed path from j to i in $G^r\}$. But, for i to be allocated a house in this round r, i would have to change its highest preference to a house owned by players in Π_i to create a cycle in this round. We now prove that all of the houses owned by players in Π_i will be retained till round k. We can prove this by induction on the distance of players from i in Π_i. Note that, if the distance is 1, their highest preference is the house currently owned by i, which won't be vacated before i's allocation. Hence, the statement is true if the distance of all players in Π_i is 1. Let us assume that it is true for some distance $d <= q$. $(q > 1)$. For all players in Π_i at distance $d = q + 1$, note that their highest preference is a house owned by a player at distance $d = q$. By our induction hypothesis, all those houses would be retained till round k. Thus, all houses owned by players at a distance $d = q + 1$ would also be retained, completing our proof.

(Efficiency) Let us assume, to the contrary, that this mechanism is not efficient. Then, there must be some mechanism which gives a better mapping a^1 (with $a^1(i)P_ia(i)$ or $a^1(i) = a(i)$ $\forall i \in N$, and $a^1! = a$). Consider both a^1 and a's mapping in the order of the TTC allocation. Let i be the first player for which $a^1(i)P_ia(i)$. Let this be round k of the TTC allocation. However, the TTC allocation guarantees the highest preference amongst M^k in round k to the participants getting a house in round k. Since this is the first person getting a different allocation in a^1, the set of houses available is M^k. Hence, there is a contradiction, which completes our proof.

■

12.2 Stable house allocation with initial endowments

A stable house allocation is one in which no group of players have a deviation with their initial endowments. The following example should clarify this concept.

12.2.1 The stable house allocation game

Consider a game with 6 players (1-6) and 6 houses $(b_1, b_2, ..., b_6)$ with the following initial endowments. $a^0(i) = b_i$ $\forall i \in \{1, 4, 5, 6\}$, $a^0(2) = b_3$ and $a^0(3) = b_2$. Suppose, $b_1P_3b_4P_3b_3...$ and $b_2P_4b_4...$ (i.e. player 3 prefers b_1 the highest, then b_4, then b_3 and so on, and player 4 prefers b_2 the highest, then b_4 and so on). Suppose, the allocation scheme is $a(i) = i$. Note that, in this scenario, players 3 and 4 have an incentive to deviate from the scheme with their initial endowments, exchanging those houses amongst themselves. (as $b_2P_4b_4$ and $b_4P_3b_3$) Thus, the allocation is not stable as $\{3, 4\}$ "block" the allocation.

12.2.2 Stable house allocation - formal definitions

Let a^0 denote the initial endowment of the players, with a^S denoting the mapping for $S \subseteq N$.

- A coalition S can block an allocation a at preference profile P if $a^S(i)P_ia(i)$ or $a^S(i) = a(i)$ $\forall i \in S$ with $a^S(i)P_ia(i)$ for atleast one $i \in S$.
- A matching a is in the core at profile P if no coalition can block a at P
- An SCF f is stable if $\forall P$, $f(P)$ is in the core at P.

Lemma 12.2 *Stability implies efficiency.*

Proof: We will prove the contrapositive of this. Suppose an allocation a is not efficient. Therefore, $S = N$ constitutes a blocking coalition. Thus, this is not stable. ■

12.2.3 Efficiency doesn't imply stability - an illustration

Consider a 3-player game (1-3) with houses (b_1, b_2, b_3). Suppose, the initial allocation is $a^0(i) = b_i$. Suppose that the preference relations are $b_1P_1b_2P_1b_3$, $b_1P_2b_2P_2b_3$ and $b_2P_3b_1P_3b_3$. Suppose, the allocation is $a(1) =$

$b_3, a(3) = b_2, a(2) = b_1$. The allocation is clearly efficient as 2,3 have their highest preference. However, the allocation is not stable as 1 can deviate and retain b_1.

12.2.4 Core matchings

Theorem 12.3 *The TTC mechanism is stable. Morover, there is a unique core matching for every preference profile.*

Proof: *(Stability)*: Suppose, TTC is not stable. Therefore, $\exists P$ (preference profile) such that matching produced by TTC is not in the core. Suppose, coalition S blocks it. Therefore, $\exists a^S$ such that $a^S(i)P_i a(i)$ or $a^S(i) = a(i)\ \forall i \in S$ with atleast one strict preference. Let $T = \{i \in S | a^S(i)P_i a(i)\}$ be the set of all strict improvement individuals. By our assumption, $T! = \Phi$. Let $S^k = S \cap \widehat{N^k}$ (where $\widehat{N^k}$ is the set of people allocated a house in round k). Note that, $\widehat{M^k}$ is the set of houses allocated in round k. Clearly, people in S^1 are getting their top-ranked houses, so they must not be in T i.e. $S^1 \subseteq (S - T)$. We will now prove, by induction, that $S^k \subseteq (S - T)$ for any round k. By our induction hypothesis, $(S^1 \cup S^2 ... S^{k-1}) \subseteq (S - T)$. Note that, TTC allocates the best houses in round k from those available (M^k). By our induction hypothesis, these are the same houses available for a^S. Hence, a^S cannot give them any better houses. Hence, $S^k \subseteq (S - T)$. Hence, $S = \cup_{k=1toK} S^k \subseteq (S - T)$. Therefore, $T = \Phi$.Hence, there is a contradiction. Therefore, TTC is stable.
(Uniqueness): Suppose, TTC returns a and $\exists a^1! = a$ which is also in core. Note that, $a(i) = a^1(i)\ \forall i \in N^1$ (as a assigns players their top preferences in round 1, and, if $a(i)! = a^1(i)$ for some $i \in N^1$, those players form a blocking coalition for a^1, leading to the contradiction that a^1 is not a core matching). We will now prove this by induction. Let us assume that $a(i) = a^1(i)\ \forall i \in \cup_{k=1toK} N^k$. We will prove that $a(i) = a^1(i)$ $\forall i \in N^{K+1}$. Note that, by our induction hypothesis, the set of houses available in round $K+1$ is the same for both allocations (M^{K+1}). Also, in a TTC allocation, the set of players being allocated in round $K+1$, N^{K+1} get their highest preference amongst the set of houses available in that round (M^{K+1}). Therefore, if $\exists i \in N^{K+1}$ such that $a(i)! = a^1(i)$, those players form a blocking coalition for a^1, leading to the contradiction that a^1 is not a core matching. This completes our proof. ■

12.3 Individual rationality

The notion of stability assumes a very high degree of cooperation between agents. There is a slightly weaker notion of individual rationality defined as follows:

Definition 12.4 *f is individually rational (IR) if, at every profile P, the matching $f(P) \equiv a$ satisfies $a(i)P_i a^0(i)$ or $a(i) = a^0(i)$. (i.e. the allocation a doesn't do worse than the initial endowment a^0 for any player.)*

Lemma 12.5 *Stability implies individual rationality.*

Proof: We will prove the contrapositive. Let us assume that some f is not individually rational. Therefore, $\exists P$ such that for some $i \in N$, $a^0(i)P_i a(i)$. This forms a blocking single agent coalition, hence, f is not stable.
■

Therefore, TTC satisfies individual rationality. The following theorem, given without proof, characterizes the TTC mechanism.

Theorem 12.6 *A one-sided matching mechanism is strategy-proof, efficient and individually rational iff it is a TTC mechanism.*

12.4 Generalized TTC mechanism

The generalized TTC mechanism mixes fixed priority and TTC by establishing a priority order for every house, which determines the initial endowment. There is a mapping $\sigma_j : M \to N \; \forall j \in M$. This might result in more than one houses being endowed to a player initially.

12.4.1 Generalized TTC- an illustration

Consider the following 4 player game. Players (1-4) are to be allocated houses from (b_1, b_2, b_3, b_4). The following are the preference orderings.

P_1	P_2	P_3	P_4
b_3	b_2	b_2	b_1
b_2	b_3	b_4	b_4
b_1	b_4	b_3	b_3
b_4	b_1	b_1	b_2

Also, $\sigma_{b_1} = \sigma_{b_2} = (1,2,3,4)$ while $\sigma_{b_3} = \sigma_{b_4} = (2,1,4,3)$.

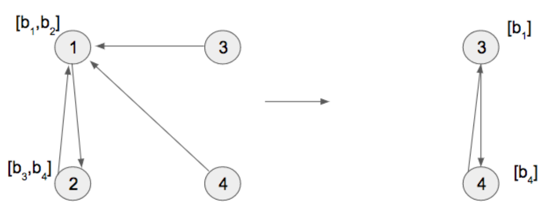

Figure 12.2: Generalized TTC illustration

This results in the following assignment. $1 \to b_3$, $2 \to b_2$, $4 \to b_1$ and $3 \to b_4$.

Theorem 12.7 *Generalized TTC is strategy-proof and efficient.*