CS698A: Selected Areas of Mechanism Design **Jan-Apr 2018**

Lecture 11: February 13, 2018

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Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor at swaprava@cse.iitk.ac.in.*

This lecture will define house allocation problems and some solution concepts.

11.1 Matching Problems

Consider the domain of problems where two collection of entities must be matched with the other (typically, each entity existing in a unique matched pair so that at-least one collection is exhausted). Assignment of project topics, finding romantic partners in a monogamous society, or being seated in a movie theatre are all good examples. In this lecture, the focus is on **One-sided Matching**, where only one collection of entities have preferences over the others. The entities having preferences are often called agents, while the other entities are called objects, leading this class of problems to be known as **Object Allocation** or **House Allocation** problems.

11.2 Recap of Social Choice Settings

Social Choice Situations employ the following notation:

- $N = \{1, ..., n\}$ A finite set of agents/players who participate in the mechanism
- X The possible set of outcomes that can be realized by the mechanism
- Θ_i Sets of "types" that the agents can take, indexed by the list of agents. The type expressed by the agent (private to the agent) is denoted by $\theta_i \in \Theta_i$. $\Theta_1 \times ... \times \Theta_n$ is often denoted as simply Θ
- $u_i : X \times \Theta_i \to \mathbb{R}$ Utility functions for each agent indexed by the set of agents
- The Social Choice Function $f : \Theta_1 \times ... \times \Theta_n \to X$ maps the true types of agents to the desirable outcome aligned with some notion of social welfare.

11.3 Voting and Gibbard-Satterthwaite Theorem

An election where a unique candidate is declared as the winner is one special example of a social choice setting, with the set of outcomes X the same as the set of candidates, while each agent's θ_i is a linear order over the set of candidates. Each agent also has a utility function $u_i : X \times \Theta_i \to \mathbb{R}$. For a more detailed description, look at [1].

Theorem 11.1 ***(Gibbard-Satterthwaite)*** *For a voting social choice setting with $|A| \geq 3$, unrestricted preferences which are linear orders, every onto and strategy-proof SCF must be dictatorial.*

11.3.1 Circumventing Gibbard-Satterthwaite

While results such as Gibbard-Satterthwaite make efficient social mechanisms seem impossible, there are several natural relaxations of the assumptions that make non-dictatorial solution concepts feasible:

- **Restricting Domain:** Instead of allowing all possible preferences, if we restrict the voters choices' to preference profiles satisfying certain properties, we might avoid the conclusion of the G-S theorem. **Example:** Voting situations with only **Single-Peaked Preferences** admit non-dictatorial strategy proof solution concepts ([2]).
- **Introducing Money:** Allowing agents to transfer utility in the form of money or a similar mechanism also prevents the conclusion of G-S theorem from holding ([3]).

11.4 One-sided Matching as a Social Choice Setting

Consider a one-sided matching with $N = \{1, ..., n\}$ the finite set of agents and $M = \{a_1, ..., a_m\}$ the finite set of objects (Assume that $m \geq n$). Every agent i has a **total order** R_i over the set of objects. In other words, for each i, R_i satisfies:

- (**Reflexivity**) $\forall a \in M(aR_ia)$
- (**Antisymmetry**) $\forall a, b \in M(aR_ib \wedge bR_ia \Rightarrow a = b)$
- (**Transitivity**) $\forall a, b, c \in M(aR_ib \wedge bR_ic \Rightarrow aR_ic)$
- (**Completeness**) $\forall a, b \in M(aR_ib \vee bR_ia)$

By P_i, we will denote the strict (irreflexive) order corresponding to R_i. Let $\mathcal{M}$ denote all possible preference orderings on M. Thus, the tuple $(P_1, ..., P_n) \in \mathcal{M}$ corresponds to a **preference profile**, with one preference relation for each agent. For $S \subseteq M$ and integer $1 \leq k \leq |S|$, $P_i(k, S) \in M$ denotes the k^{th} most preferred object from S when ordered by P_i.

11.4.1 Departure from the G-S Setting

Note that in the voting social choice setting, the set of outcomes X was taken as the set of objects (candidates) M, ensuring that each preference order P_i continued to be a total order over the outcome set. Indeed, the outcome set consisting of allocations instead of objects becomes the reason we are able to talk of non-dictatorial solution concepts for one-sided matching settings. Consider A to be the set of all allocations (**injective mappings**) $a : N \to M$. For $a, a' \in A$, (an appropriate extension of) P_i can no longer distinguish between a and a' if $a(i) = a'(i)$, even though a might not be equal to a'.

11.4.2 Desirable Properties of Solution Concepts

Let $f : \mathcal{M}^n \to A$ be a SCF. The following are some desirable properties for f in the house-allocation setting:

Definition 11.2 *f is* ***strategy-proof*** *if for each agent i:*

$$\forall P_i, P_i' \in \mathcal{M}, \forall P_{-i} \in \mathcal{M}^{n-1}(([f(P_i, P_{-i})(i)]P_i[f(P_i', P_{-i})(i)]) \vee ([f(P_i, P_{-i})(i)] = [f(P_i', P_{-i})(i)]))$$

Definition 11.3 *f is **efficient** if for each $P \in \mathcal{M}$ and $a \in A$, if there exists another $a' \in A$ ($a \neq a'$) such that $\forall i \in N(a'(i)P_i a(i) \vee a'(i) = a(i))$, then $f(P) \neq a$.*

11.4.3 Serial Dictatorships

Consider a fixed bijective map (permutation) $\sigma : N \to N$. A **serial dictatorship** based on σ allocates (defined recursively) player $\sigma(i)$ the object $P_{\sigma(i)}(1, M \backslash \{a(\sigma(1)), ..., a(\sigma(i-1))\})$. In other words, players are ordered by the permutation σ and allowed to pick the object they prefer most among those available. Serial Dictatorships are also known as **fixed priority mechanisms**.

Theorem 11.4 *Every serial dictatorship is strategy-proof and efficient.*

Proof: Since σ is decided a priori and is not affected by the R_i's that players report, a player is always presented by the same set of alternatives keeping the other players' preferences (P_{-i}) constant. In choosing between a fixed set of objects, he/she can gain no advantage by revealing a false preference. Similarly, if there existed an allocation where one of the players had a more preferred alternative available from the fixed set of alternatives presented to him (constant since all players earlier must have been allocated the same objects by the conditions in the premise of efficiency), he/she would pick that alternative instead, leading to a contradiction. Thus, serial dictatorships are both strategy-proof and efficient. ∎

11.4.4 Other Strategy Proof Allocations

Serial dictatorships are not the only strategy-proof and efficient solution concepts that house-allocation problems admit. Consider for example, the problem when there are three agents $N = \{1, 2, 3\}$, three objects $M = \{a_1, a_2, a_3\}$, and a modified serial dictatorship with

$$\sigma = \begin{cases} (1,2,3) & \text{if } P_1(1) = a_1; \\ (2,1,3) & \text{otherwise.} \end{cases}$$

Going through why this would indeed be both strategy-proof and efficient is an instructive exercise.

11.5 Top Trading Cycle with Fixed Endowments

A particularly effective truthful solution concept to the house-allocation problem is the top trading cycle method with fixed endowments, a technique outlined by Gale in [4]. This lecture contains an intuitive example of how it works without going into formal details.

11.5.1 An Illustrative Example

Consider agent set $N = \{1, 2, 3, 4, 5, 6\}$, choosing between houses $M = \{a_1, a_2, a_3, a_4, a_5, a_6\}$. Their preference profile is given in Table 11.1. Assign each player a random house to begin with (the "fixed endowment"). The agents can then be represented as a digraph with each node pointing to the agent that contains its most preferred house. Since each node has exactly one outgoing edge, the digraph must contain a cycle (think why!). Pick an arbitrary cycle (in case there are more than 1), shuffle the house allocation cyclically so each agent in the cycle receives their most preferred house, and then delete the agents of the cycle and the houses they were allocated from the problem till no agent remains. An illustration of this procedure is outlined in the following tables and diagrams:

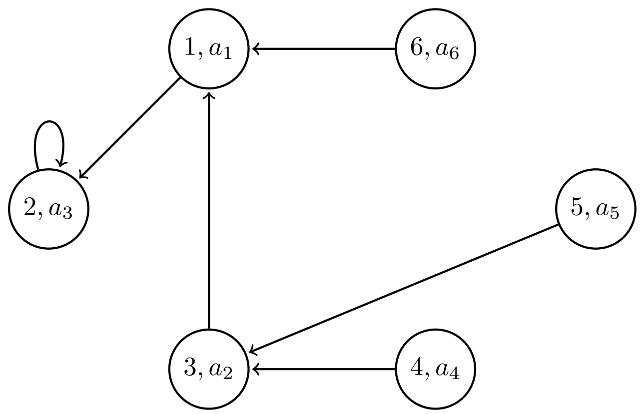

Figure 11.1: Initial Preference Graph

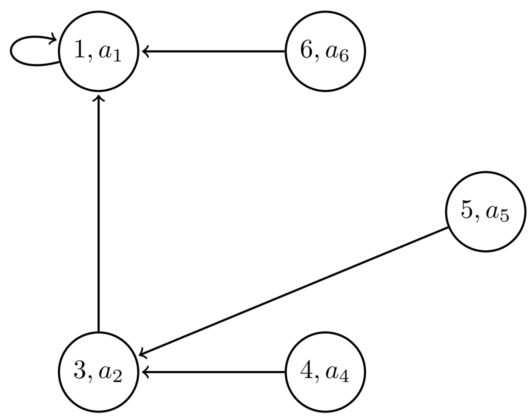

Figure 11.2: After removal of 2

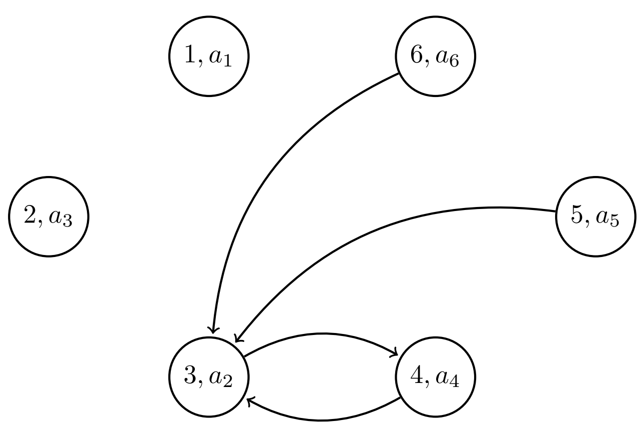

Figure 11.3: After removal of 2,1

Table 11.1: Initial House Preferences for Agents

1	**2**	**3**	**4**	**5**	**6**
a_3	a_3	a_1	a_2	a_2	a_1
a_1	a_2	a_4	a_1	a_1	a_3
a_2	a_1	a_3	a_5	a_6	a_2
a_4	a_5	a_2	a_4	a_4	a_4
a_5	a_4	a_6	a_3	a_5	a_6
a_6	a_6	a_5	a_6	a_3	a_5

Table 11.2: House Preferences for Agents after removal of 2

1	**2**	**3**	**4**	**5**	**6**
$\cancel{a_3}$	a_3	a_1	a_2	a_2	a_1
a_1	$\cancel{a_2}$	a_4	a_1	a_1	$\cancel{a_3}$
a_2	$\cancel{a_1}$	$\cancel{a_3}$	a_5	a_6	a_2
a_4	$\cancel{a_5}$	a_2	a_4	a_4	a_4
a_5	$\cancel{a_4}$	a_6	$\cancel{a_3}$	a_5	a_6
a_6	$\cancel{a_6}$	a_5	a_6	$\cancel{a_3}$	a_5

Table 11.3: House Preferences for Agents after removal of 2,1

1	**2**	**3**	**4**	**5**	**6**
$\cancel{a_3}$	a_3	$\cancel{a_1}$	a_2	a_2	$\cancel{a_1}$
a_1	$\cancel{a_2}$	a_4	$\cancel{a_1}$	$\cancel{a_1}$	$\cancel{a_3}$
$\cancel{a_2}$	$\cancel{a_1}$	$\cancel{a_3}$	a_5	a_6	a_2
$\cancel{a_4}$	$\cancel{a_5}$	a_2	a_4	a_4	a_4
$\cancel{a_5}$	$\cancel{a_4}$	a_6	$\cancel{a_3}$	a_5	a_6
$\cancel{a_6}$	$\cancel{a_6}$	a_5	a_6	$\cancel{a_3}$	a_5

Table 11.4: House Preferences for Agents after removal of 2,1,3,4

1	**2**	**3**	**4**	**5**	**6**
$\cancel{a_3}$	a_3	$\cancel{a_1}$	a_2	$\cancel{a_2}$	$\cancel{a_1}$
a_1	$\cancel{a_2}$	a_4	$\cancel{a_1}$	$\cancel{a_1}$	$\cancel{a_3}$
$\cancel{a_2}$	$\cancel{a_1}$	$\cancel{a_3}$	$\cancel{a_5}$	a_6	$\cancel{a_2}$
$\cancel{a_4}$	$\cancel{a_5}$	$\cancel{a_2}$	$\cancel{a_4}$	$\cancel{a_4}$	$\cancel{a_4}$
$\cancel{a_5}$	$\cancel{a_4}$	$\cancel{a_6}$	$\cancel{a_3}$	a_5	a_6
$\cancel{a_6}$	$\cancel{a_6}$	$\cancel{a_5}$	$\cancel{a_6}$	$\cancel{a_3}$	a_5

References

[1] Allan Gibbard. Manipulation of voting schemes: A general result. *Econometrica*, 41(4):587–601, 1973.

[2] Hervé Moulin. On strategy-proofness and single peakedness. *Public Choice*, 35(4):437–455, 1980.

Table 11.5: Final House Preferences and Allocations for Agents

1	**2**	**3**	**4**	**5**	**6**
$\cancel{a_3}$	a_3	$\cancel{a_1}$	a_2	$\cancel{a_2}$	$\cancel{a_1}$
a_1	$\cancel{a_2}$	a_4	$\cancel{a_1}$	$\cancel{a_1}$	$\cancel{a_3}$
$\cancel{a_2}$	$\cancel{a_1}$	$\cancel{a_3}$	$\cancel{a_5}$	$\cancel{a_6}$	$\cancel{a_2}$
$\cancel{a_4}$	$\cancel{a_5}$	$\cancel{a_2}$	$\cancel{a_4}$	$\cancel{a_4}$	$\cancel{a_4}$
$\cancel{a_5}$	$\cancel{a_4}$	$\cancel{a_6}$	$\cancel{a_3}$	a_5	a_6
$\cancel{a_6}$	$\cancel{a_6}$	$\cancel{a_5}$	$\cancel{a_6}$	$\cancel{a_3}$	$\cancel{a_5}$

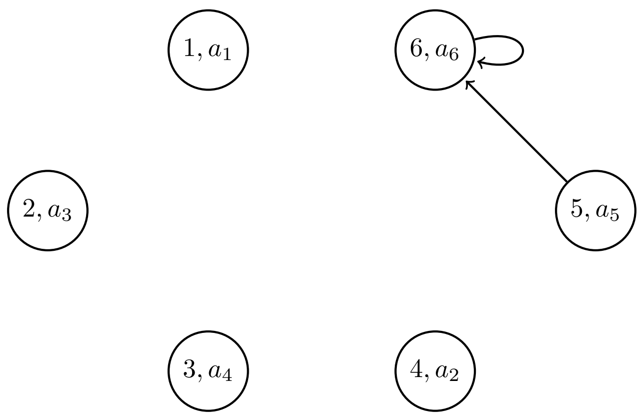

Figure 11.4: After removal of 2,1,3,4

[3] William Vickrey. Counterspeculation, auctions, and competitive sealed tenders. *The Journal of Finance*, 16(1):8–37, 1961.

[4] Lloyd Shapley and Herbert Scarf. On cores and indivisibility. *Journal of Mathematical Economics*, 1(1):23 – 37, 1974.