CS-698A: Selected Areas of Mechanism Design **Jan-Apr 2018**

Lecture 9: Uniqueness of Shapley Value

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Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor.* s lecture's notes illustrate some uses of various LaTeX macros. Take a look at this and imitate.

9.1 Carrier Game

A coalition is winning if it contains a distinguished set, say T, is an influential coalition. Containing it can pass a bill or do a change.

Definition 9.1 *Let $T \subseteq N$ be a non-empty coalition. The carrier game over T is the game (N, ϑ) such that for each coalition $S \subseteq N$,*

$$u_T(S) = \begin{cases} 1 & \textit{if } T \subseteq S \\ 0 & \textit{if otherwise} \end{cases}$$

Theorem 9.2 *Every game (N, ϑ) is a linear combination of carrier games.*

Proof: To define any TU game , we need to define the valuations over all non-empty subsets. Hence, every game (N, ϑ) is a point in $\mathbb{R}^{2^n-1}$. We have to show that carrier games span this space. We have to find carrier games that are linearly independent and form a basis.
Suppose , carrier games are linearly dependent (for contradiction). $\exists$ real numbers $\{\alpha_T\}_{\{T\subseteq N, T\neq\phi\}}$, not all zero, such that $\sum_{\{\ T\subseteq N, T\neq\phi\}} \alpha_T \ u_T(S) = 0, \forall\ S \subseteq N$.
Let $\mathcal{T} = \{T \subseteq N : T \neq \phi, \alpha_T \neq 0\}$ collections of non empty coalitions with non zero coefficients in the above equation. Since $\{\alpha_T\}_{\{T\subseteq N, T\neq\phi\}}$ are not all zero, there exists a minimal coalition in $\mathcal{T}$, i.e. coalition with smallest cardinality . Say $S_0 \in \mathcal{T}$ is one such coalition . $\nexists$ any subset of S_0 with positive coefficients.
Consider $\sum_{\{\ T\subseteq N, T\neq\phi\}} \alpha_T\ u_T(S_0) = \sum_{\{\ T\subseteq S_0, T\neq\phi\}} \alpha_T\ u_T(S_0) + \alpha_{S_0}\ u_{S_0}(S_0) + \sum_{\{\ T\not\subseteq S_0\}} \alpha_T\ u_T(S_0) = \alpha_{S_0} \neq 0$.

■

Theorem 9.3 *Let T be a non-empty coalition , and $\alpha \in \mathbb{R}$. Define a game $(N, u_{T,\alpha})$ as follows,*

$$u_{T,\alpha}(S) = \begin{cases} \alpha & \textit{if } T \subseteq S \\ 0 & \textit{if otherwise} \end{cases}$$ *If ϕ is a solution concept that satisfies efficiency, symmetry, null player property, then*

$$\Phi_i(N, u_{T,\alpha}) = \begin{cases} \frac{\alpha}{|T|} & \textit{if } i \in T & \dots(1) \\ 0 & \textit{if otherwise} & \dots(2) \end{cases}$$

Observation 1: *$\imath \notin T$ is a null player.*
Observation 2: *all $i, j \in T$ are symmetric together with efficiency (1) follows.*

9.2 Uniqueness of Shapley Value

Proof: The Shapley Value satisfies the four poperties. We need to show that any Φ satisfying these four properties is identical to Sh. Theorem 1 says that for any game (N, ϑ), We can write ϑ as sum of $u_{T,\alpha}$'s Let us pick (N, ϑ) $\exists$ real numbers $\{\alpha_T\}_{\{T \subseteq N, T \neq \phi\}}$ such that
$\vartheta(S) = \sum_{\{\ T \subseteq N, T \neq \phi\}} \mathrm{u}_{T,\alpha_T}(S)$ Theorem 2 says,since both Φ and Sh satisfies efficiency, symmetry and null player property

$$\Phi(N, u_{T,\alpha_T}) ,= Sh(N, \alpha_T)\ ,\ \forall T \subseteq N,\ T \neq \phi.$$

Since both Φ and Sh satisfy additivity

$$\Phi(N, u_{T,\vartheta}) = \sum_{\{\ T \subseteq N, T \neq \phi\}=} \Phi(N, u_{T,\alpha_T}) = \sum_{\{\ T \subseteq N, T \neq \phi\}} \mathrm{Sh}(\mathrm{N},\alpha_T) = Sh(N, \vartheta)$$ We started

with an arbitrary game (N, ϑ), hence this holds for all such games. ■

9.2.1 Examples

1. *Two Player Bargaining*
   $\vartheta(1) = \vartheta(2) = 0$, $\vartheta(1, 2) = 1$
   Player 1 2 are symmetric and Shapley value is efficient.

   $$Sh(N, \vartheta) = (\tfrac{1}{2}, \tfrac{1}{2})$$

2. *Majority Game*

   $$\vartheta(S) = \begin{cases} 0 & \text{if } |S| \leq \frac{n}{2} \\ 0 & \text{if } |S| > \frac{n}{2} \end{cases}$$

   All players are symmetric, hence Shapley Value are same, together with efficiency.

   $$Sh(N, \vartheta) = (\tfrac{1}{n}, \cdots, \tfrac{1}{n})$$

3. *Gloves Game*
   $\vartheta(1) = \vartheta(2) = \vartheta(3) = \vartheta(1, 2) = 0$
   $\vartheta(1, 3) = \vartheta(2, 3) = \vartheta(1, 2, 3) = 1$

Permutation	Player 1	Player 2	Player 3
1, 2, 3	0	0	1
1, 3, 2	0	0	1
2, 1, 3	0	0	1
3, 1, 2	1	0	0
3, 2, 1	0	1	0
	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{4}{6}$

Entries in the cells of above matrix is calculated by the formula,

$$\vartheta(P_j(\pi) \cup \{i\}) - \vartheta(P_i(\pi)), \text{ for every player i}$$

$$Sh(N, \vartheta) = (\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{2}{63})$$

The Shapley Value emphasizes that player 3 is the most powerful player. But players 1 2 do not get zero in the allocation.
Core: (0, 0, 1) is the singleton.
Hence, Shapley Value is not in core.

9.3 Application: Shapley - Shubik Power Index

Definition 9.4 *Simple Games:-*
The value of any coalition can either be 0 or 1.

Definition 9.5 *Monotone Games:-*
If any coalition has value 1, every superset of that coalition also has value 1.

Definition 9.6 *Shapley - Shubik Power Index*
The Shapley - Shubik Power Index is a function associating each simple monotonic game with its Shapley Value. The i^{th} co-ordinate denotes the "Power" of player i in this game.

$$Sh_i(N, \vartheta) = \sum_{\{\ S \subseteq N - \{i\} : S \cup \{i\} \text{ is winning, and } S \text{ is losing}\}} A(S)$$

where $A(S) = |S|! * (n - |S| - 1)!/n!$

counting all such scenarios where player i is pivotal.

9.3.1 Case Study: UN Security Council

UN body of international political system established in 1945 after world war II. Till 1965 there were 5 Permanent members and 6 Non-Permanent members. A resolution was adopted that if it receives at least 7 votes but all Permanent members have Veto Power.

It was debated about the unequal distribution of power in the security council. After 1965 there were 5 Permanent members and 10 Non-Permanent members. A resolution needed 9 votes but Veto Power remained with the Permanent members.
This is a simple monotonic game. Let us compute the Shapley - Shubik Power Index. Solution

P: Permanent Members, NP: Non-Permanent Members Pre-1965:

$$\vartheta(S) = \begin{cases} 1 & \text{if } P \subseteq S and |S| \geq 7 \\ 0 & \text{if otherwise} \end{cases}$$

For NP i, $Sh_i(N, \vartheta) = \binom{5}{1}$ * 6!4!/11! = 1/462
All NP are Symmetric and all P are symmetric. Shapley Value is efficient, hence for a permanent j,
$Sh_j(N, \vartheta)$ = 1/5(1 - 6/462) = 91.2/462
power ratio of NP to P = 1:91.2

Post-1965:

$$\vartheta(S) = \begin{cases} 1 & \text{if } P \subseteq S and |S| \geq 9 \\ 0 & \text{if otherwise} \end{cases}$$

For NP i, $Sh_i(N, \vartheta) = \binom{9}{3}$ * 8!6!/15! = 4/2145

Shapley Value is efficient,
For P j, $Sh_j(N, \vartheta)$ = 1/5(1 - 10 * 4/2145) = 421/2145
power ratio of NP to P = 1: 105.25
Restructuring actually increased the power of the Permanent Members.

9.4 Convex games

Theorem 9.7 *If (N, ϑ) is a Convex game, Shapley Value is in the Core.*

Proof: For any permutation $\pi \in \Pi(N)$, consider the imputation w^π,
$w^\pi{}_1 = \vartheta(P_j(\pi) \cup \{i\}) - \vartheta(P_i(\pi))$, For all $i \in N$
w^π is in the core $\forall \pi \in \Pi(N)$. Since core is convex, any convex combination of these points will be in core.
Shapley Value is one such convex combination. ■