CS-698A: Selected Areas of Mechanism Design **Jan-Apr 2018**

Lecture 8: Limitations of core - I

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8.1 Recap

In the previous lecture we saw the consistency property of the core. We then reviewed Davis-Maschler reduced game and saw that the core satisfied the Davis-Maschler reduced game property. Towards the end we saw that convex games have non-empty core.

Shapley Value

- It is a single valued solution concept
- It is based on axioms and can be considered similar to Nash Bargaining

Notation:
We use ϕ to denote single valued solution.
$\phi_i(N, v)$ is the allocation to player i $\in N$

8.2 Axioms of Shapley Value

The four axioms are as follows:

1. **Efficiency**: If for all TU games (N,v), it holds that $\sum_{i \in N} \phi_i(N, v) = v(N)$ we say ϕ satisfies efficiency
2. **Symmetry**: Two players i and j are symmetric if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$

   Definition: ϕ satisfies symmetry if $\phi_i(N, v) = \phi_j(N, v) \forall i, j$ (both being symmetric players) and $\forall (N, v)$. This can be thought of as giving equal treatment to equals.
3. **Null Player Property**: Player i is a null player in (N,v) if for every $S \subseteq N, v(S \cup \{i\}) = v(S)$. It is evident that $v(i) = 0$ for a null player as we can take S as an empty set.

   Definition: If $\phi_i()N, v = 0 \forall (N, v)$ and for all null players i then ϕ satisfies null player property.
4. **Additivity**: ϕ is said to be satisfying additivity if for every pair of coalitional games (N,v) and (N,w), we have $\phi(N, v + w) = \phi(N, v) + \phi(N, w)$.
   The above equation tells us that reward from playing two different games independently is same as playing one with added valuation.

8.2.1 Some examples

We will be seeing few examples to gain clarity about the four axioms.

1. $\Psi_i(N, v) = v(i)$
   We check for the four axioms one by one.

   - Null player: As for null player $v(i) = 0$ we have $\Psi_i(N, v) = 0$
   - Additivity: $\Psi_i(N, v + w) = (v + w)(i) = v(i) + w(i) = \Psi_i(N, v) + \Psi_i(N, v)$
   - Symmetry: If $S \subseteq N \setminus \{i, j\}$ we have $\check{(}S \cup \{i\}) = v(S \cup \{j\})$, now taking S to be an empty set i.e. $S = \phi \Rightarrow$v(i)=v(j) $\Rightarrow \Psi_i(N, v) = \Psi_j(N, w)$
   - Efficiency: it is possible to find examples where $\sum v(i) \neq v(N)$, hence efficiency is not necessary.

2. A player is said to be a dummy player when it satisfies $v(S \cup \{i\}) = v(S) + v(i) \forall S \setminus \{i\}$
   It can be easily concluded that a null player is always a dummy player. Now (N, v) has $d(v)$ number of dummy players.
   Let us look at the following solution concept:

$$\Psi_i(N, v) = \begin{cases} v(i) + \frac{v(N) - \sum_{j \in N}}{n - d(v)} & \text{;i is not a dummy player} \\ v(i) & \text{;i is a dummy player} \end{cases}$$

   - Efficiency and Null player property are satisfied and can be observed easily, also null player is a dummy and gets zero.
   - Symmetry: When both players are either dummy or not dummy then this is true. However, we would like to see if they are symmetric if one(say i) is not dummy and other(say j) is a dummy.
     From the arguments of symmetry we have, $v(S \cup \{i\}) = v(S \cup \{j\}) \; \forall S \subseteq N \setminus \{i, j\}$
     $v(S \cup \{i\}) = v(S \cup \{j\})$
     $= v(S) + v(j)$
     $= v(S) + v(i) \quad \{whenS = \phi, \; v(i) = v(j)\}$
     $Adding \; v(j) \; on \; both \; sides, \; we \; get$
     $\Rightarrow v(S \cup \{i\}) + v(j) = v(S) + v(i) + v(j)$
     $\Rightarrow v(S \cup \{j\} \cup \{i\}) = v(S \cup \{j\}) + v(i)$
     $\Rightarrow v(\bar{S} \cup \{i\}) = v(\bar{S}) + v(i) \quad \forall \; \bar{S} \subseteq N \setminus \{i, j\}$
     We thus find that i is a dummy player which contradicts the initial assumption we made.
     Consider two games,
     Game 1: v(1) =v(2) =v(3) =v(1,2) =v(1,3)=0 & v(2,3)=v(1,2,3)=1
     Game 2: u(1) =u(2) =u(3) =u(1,3)=0 & u(1,2)=u(2,3)=u(1,2,3)=1
     In Game 1, player 1 is a null player and therefore, a dummy player. We have the following solution concept from the two games and their additive counterpart:
     $\Psi(N, v) = (0, \frac{1}{2}, \frac{1}{2}) \quad \Psi(N, u) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ while $\Psi(N, v + u) = (\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$
     We can note that however this solution concept is not additive, it satisfies the rest of the three axioms

3. $\Psi_i(N, v) = \; max_{\{S : i \notin S\}}(v(S \cup \{i\}) - v(S))$
   It can be shown that this solution concept is symmetric and null but neither efficient nor additive.The proof of which has been left as an exercise.

4. $\Psi_i(N, v) = v(1, 2, \ldots, n) - v(1, 2, \ldots, n - 1)$
   The above solution satisfies efficiency, null player property and additivity but not symmetry.Consider the example v(i)=v(j)=0 and v(1,2)=1, it is evident that this doesn't satisfy symmetry.

8.2.2 Generalizing for any permutation

In this section we look forward to generalize the solution concept for any order of players and not just the identity one. Let $\Pi(N)$ be set of all possible permutation of N players.Also, we have $\pi \in \Pi(N)$ as one of the permuted set of players.

Predecessor set of i in $\pi \in \Pi(N)$, hence $P_i(\pi) = \{j \in N \; \pi(j) < \pi(i)\}$
Now, $P_i(\pi) = \phi$ if $\pi(i) = 1$
$P_i(\pi) \cup \{i\} = P_k(\pi) \iff \pi(k) = \pi(i) + 1$
We can now define the solution concept as $\Psi_i(N, v) = v(P_i(\pi) \cup \{i\}) - v(P_i(\pi))$

Note: We can observe from example 4 that this category of solution concept satisfies efficiency,null player property and additivity but not symmetry.

8.3 Shapley Value

There is a unique solution concept which satisfies all of the four axioms mentioned before in this lecture.
Definition:(Shapley 1953) The Shapley value is a solution concept denoted as Sh and defined as

$$Sh_i(N, v) = \frac{1}{n!} \sum_{\{\pi \in \Pi(N)\}} \Psi_i^\pi(N, v) \quad \forall i \in N$$

This can be thought of as a simple average over all Ψ_i^πs

Theorem 8.1 *The Shapley value is the only single valued solution concept which satisfies efficiency, null player property, symmetry and additivity.*

An equivalent formula for Shapley value is:
$Sh_i(N, v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} (v(P_i(\pi) \cup \{i\}) - v(P_i(\pi))$

$= \frac{1}{n!} \sum_{S \subseteq N \setminus \{i\}} \sum_{\substack{\pi \in \Pi(N) \\ P_i(\pi) = S}} (v(S \cup \{i\}) - v(S))$
$= \sum_{S \subseteq N \setminus \{i\}} \frac{\|S\|!(n - \|S\| - 1)!\{v(S \cup \{i\}) - v(S)\}}{n!}$

Interpretation: The above expression is the average marginal contribution to all the other coalitions.

Proof: Part 1 - To prove that Shapley value satisfies all four axioms.
We know by that each of Ψ_i^π satisfies the four axioms and thus their average.
Symmetry: Given i and j to be two symmetric players and a permutation π, we define a function f s.t.
f: $\pi \to f(\pi)$ (f(π) swaps the position of i and j)

$$(f(\pi))(k) = \begin{cases} \pi(j) & \text{if k=i} \\ \pi(i) & \text{if k=j} \\ \pi(k) & \text{if k} \neq i, j \end{cases}$$

It is evident that f is a bijection.

Claim: $\Psi_i^\pi(N, v) = \Psi_j^{f(\pi)}(N, v)$
$\iff$v(P$_i(\pi) \cup \{i\}) - v(P_i(\pi) = v(P_j(f(\pi)) \cup \{j\}) - v(P_j(f(\pi))$ ...1

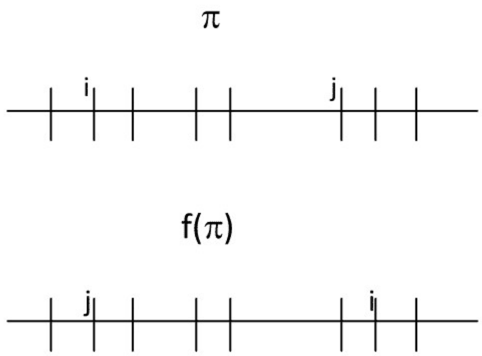

Figure 8.1: Case 1

Here we can have two cases, either player i comes before player j or vice versa.
Case 1: Player i comes before player j in π , i.e $j \notin P_i(\pi)$
It is clear that $P_j(f(\pi)) = P_i(\pi) \Rightarrow v(P_j(f(\pi))) = v(P_i(\pi))$ Also, given that i and j being symmetric,

$$v(P_i(\pi) \cup \{i\}) = v(P_j(f(\pi)) \cup \{j\})$$

Hence, we conclude that equation 1 holds.

Case 2: Player i appears after j in π , i.e. $j \in P_i(\pi)$
$\Rightarrow P_i(\pi) \cup \{i\} = P_j(f(\pi) \cup \{j\})$
$\Rightarrow v(P_i(\pi) \cup \{i\}) = v(P_j(f(\pi) \cup \{j\}))$
Also,we have $P_i(\pi) \setminus \{j\} = P_j(f(\pi)) \setminus \{i\}$
Since i and j are symmetric,
$v(P_i(\pi)) = v(P_j(f(\pi)))$

Thus, equation 1 holds. ■

References

[MSZ] Michael Maschler, Eilon Solan, Shmuel Zamir, Game Theory, cambridge university press