Lecture 7: Consistency Property of the Core and Convex Games

Lecturer: Swaprava Nath *Scribe(s): Shubham Kumar Nigam*

Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor.*

7.1 The Consistency property of the Core

Consider the three-player coalitional game:
$v(1) = v(2) = v(3) = 0, v(1, 2) = v(2, 3) = 1, v(1, 3) = 2, v(1, 2, 3) = 3.$
The imputation $(2, 0.5, 0.5)$ is in the core of this game. Suppose that the players decide to divide the worth of the grand coalition, 3 on the basis of this vector. Suppose now that Player 3 leaves with his share 0.5. Do the other two players can divide the remaining worth of 2.5 in a better way? For example the division can be $(1.25, 1.25)$, which is not in core. To answer this question, we will attempt to describe the new situation between players $\{1, 2\}$ as a new game. What is this new game? What is its core? One way to define the new game is as follows:

7.1.1 Davis-Maschler reduced game

Definition 7.1 ***Reduced game:*** *Let (N, v) be a coalitional game, let S be a nonempty coalition, and let x be an efficient vector in $\mathbb{R}^N$ and hence $x(N) = v(N)$. The Davis-Maschler reduced game to S relative to x, denoted by (S, w_S^x), is the coalitional game with the set of players S and a coalitional function w_S^x,*

$$w_S^x = \begin{cases} \max\limits_{Q \subseteq N \setminus S} [v(R \cup Q) - x(Q)] & \text{if } R \neq \phi \text{ and } R \subset S \\ x(S) & \text{if } R = S \\ 0 & \text{if } R = \phi \end{cases}$$

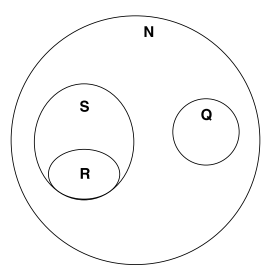

Figure 7.1: figure for definition of w_S^x if $R \neq \phi$ and $R \subset S$

Definition 7.2 *A set solution concept ϕ satisfies the Davis-Maschler reduced game property if for every coalitional game (N, v), for every nonempty coalition $S \subseteq N$, and for every vector $x \in \phi(N, v)$, $(x_i)_{i \in S} \in \phi(S, w_S^x)$.*

The reduced game property is a consistency property as, if the players believe in the solution concept ϕ, then every set of players S considering redistributing $\sum_{i \in S} x_i$ among its members will refrain from doing so, because the vector $(x_i)_{i \in S}$ is in the solution ϕ of the game reduced to S.

Theorem 7.3 *The core satisfies the DavisMaschler reduced game property.*
Proof: *Let x be a point in the core of the coalitional game (N, v), and let S be a nonempty coalition.*
To show: *$(x_i)_{i \in S}$ is in core of (S, w_S^x), for which we need to show,*

1. *$x(R) \geq w_S^x(R)$ for every $R \neq \phi$ and $R \subset S$*
2. *$w_S^x(S) = x(S) = \sum_{i \in S} x_i$.*

(2) *is immediate from the definition of w_S^x.*
For (1), let $R \subset S$ be a nonempty coalition. We want to show that $x(R) \geq w_S^x(R)$. By the definition of $w_S^x(R)$, there exists a coalition $Q \subseteq N \setminus S$ such that $w_S^x(R) = v(R \cup Q) - x(Q)$. Then we have

$$w_S^x(R) = v(R \cup Q) - x(Q) = v(R \cup Q) - x(R \cup Q) + x(R)$$

The vector x is in the core of (N, v), hence $x(R \cup Q) \geq v(R \cup Q)$, and therefore

$$x(R) \geq w_S^x(R),$$

which is what we wanted to prove. ■

7.2 Convex games revisited

Definition 7.4 ***Convex games:*** *A coalitional game (N, v) is convex if for every pair of coalitions S and T the following holds:*

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$$

Remark: Every subgame (S, v) of a convex game is convex.
Proof Exercise: If (N, v) is a convex game then for every coalition $S \subseteq N$, the subgame (S, v), restricted to the players in S and v restricted to the power set of S, is also a convex game.
Convex games are characterized by the property that players have an incentive to join large coalitions. Formally, this can be stated as:

Theorem 7.5 *For any coalitional game (N, v) the following conditions are equivalent:*

1. *(N, v) is a convex game.*
2. *For every $S \subseteq T \subseteq N$ and every $R \subseteq N \setminus T$,*

$$v(S \cup R) - v(S) \leq v(T \cup R) - v(T).$$

3. *For every $S \subseteq T \subseteq N$ and every $i \in N \setminus T$,*

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T).$$

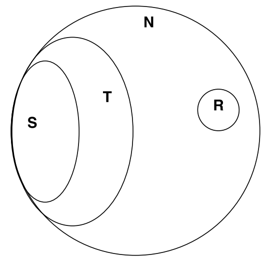

Figure 7.2: Figure for condition statement (3) of theorem(7.5)

Proof: *Part(1.) At first we will prove that* (1) *implies* (2). *Suppose* (N, v) *is a convex game and* S *and* T *are two coalitions satisfying* $S \subseteq T \subseteq N$ *, and that* $R \subseteq N \setminus T$ *. By* (1) *the game is convex, and therefore,*

$$v(S \cup R) + v(T) \leq v(S \cup T \cup R) + v((S \cup R) \cap T).$$

but, $S \cup T \cup R = T \cup R$.

$$v(S \cup R) + v(T) \leq v(T \cup R) + v((S \cap T) \cup (R \cap T)).$$

$S \cap T = S$ *and* $R \cap T = \phi$.

$$v(S \cup R) + v(T) \leq v(T \cup R) + v(S \cup \phi).$$

$$v(S \cup R) + v(T) \leq v(T \cup R) + v(S).$$

by, rearranging the terms,

$$v(S \cup R) - v(S) \leq v(T \cup R) - v(T).$$

Part(2.) If set $R = \{i\}$, (2) *implies* (3).

Part(3.) The left part is to prove (3) *implies* (1). *Let* C *and* D *be two coalitions and* C *is not contained in* D. *Define* $A := C \cap D$ *and* $B := C \setminus D$ *(figure 7.3).*

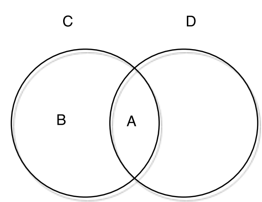

Figure 7.3: Figure used in part(3) of proof of theorem(7.5)

Since C *is not contained in* D *, the set* B *is nonempty. Let* $B = \{i_1, i_2, ..., i_k\}$
Since $A \subseteq D$, $A \cup \{i_1, ..., i_l\} \subseteq D \cup \{i_1, ..., i_l\}$ *for every* $l = 0, 1, ..., k-1$. *Moreover,* $i_{l+1} \notin D \cup \{i_1, ..., i_l\}$.
By (3), *for every* $l = 0, 1, ..., k-1$,

$$v(A \cup \{i_1, ..., i_l, i_{l+1}\}) - v(A \cup \{i_1, ..., i_l\}) \leq v(D \cup \{i_1, ..., i_l, i_{l+1}\}) - v(D \cup \{i_1, ..., i_l\}).$$

by adding the inequalities for $l = 0, 1, ..., k-1$,

$$v(A \cup B) - v(A) \leq v(D \cup B) - v(D).$$

$A \cup B = C$, $A = C \cap D$ and $D \cup B = C \cup D$. Therefore,

$$v(C) - v(C \cap D) \leq v(C \cup D) - v(D).$$

or,

$$v(C \cup D) + v(C \cap D) \geq v(D) + v(C) = (1).$$

Since this inequality holds for every two coalitions C and D , the game is convex. This concludes the proof of Theorem (7.5). ■

Theorem 7.6 *Convex games have non-empty core.*
Proof: *Let (N, v) a convex and definite x as:*
$x_1 = v(1)$
$x_2 = v(1,2) - v(1)$
$x_3 = v(1,2,3) - v(1,2)$
...
$x_N = v(1,2,3,...,N) - v(1,2,3,...N-1)$

Observation: x is an efficient vector:

$$x(N) = \sum_{i \in N} x_i = v(1) + v(1,2) - v(1) + v(1,2,3) - v(1,2)...v(1,2,3,...,N) - v(1,2,3,...N-1) = v(N)$$

To Show: $x(S) \geq v(S)$, $\forall S \subseteq N$
Let $S = \{i_1, i_2, ..., i_k\}$ be a coalition. Without loss of generality let us assume that, $i_1 < i_2 < i_3 < ... < i_k$. Then, $\{i_1, i_2, ..., i_{j-1}\} \subseteq \{1, 2, ..., i_j - 1\}$ for every $j \in \{1, 2, ..., k\}$. Since, (N, v) is complex game, implication (3) *of theorem(7.5) gives,*

$$v(1,2,...,i_j) - v(1,2,...,i_j - 1) \geq v(i_1, i_2, ..., i_j) - v(i_1, i_2, ..., i_{j-1})$$

Hence,

$$x(S) = \sum_{j=1}^{k} x_{ij} = v(1,2,...,i_1) - v(1,2,...,i_1 - 1) + v(1,2,...,i_2) - v(1,2,...,i_2 - 1) + ... + v(1,2,...,i_k) - v(1,2,...,i_k - 1)$$

$$\geq v(i_1) - v(\phi) + v(i_1, i_2) - v(i_1) + ... + v(i_1, i_2, ..., i_k) - v(i_1, i_2, ..., i_{k-1}) = v(i_1, i_2, ..., i_k) = v(S)$$

■

Remark: In the proof of theorem (7.6), we proved that $(v(1), v(1,2) - v(1), ..., v(1,2,...,n) - v(1,2,...,n1))$ is an imputation in the core of the coalitional game (N, v). In that case, we considered the players according to the ordering $1, 2, ..., n$. But the same result obtains under any ordering of the players. In other words, given any ordering $= (i_1, i_2, ..., i_n)$ of the players, the following is an imputation in the core of the game (N, v):
$w^{\pi} := (v(i_1), v(i_1, i_2) - v(i_1), v(i_1, i_2, i_3) - v(i_1, i_2), ..., v(N) - v(N \; \{i_n\}))$
This imputation corresponds to the following description: the players enter a room one after the other, according to the ordering π. Each player receives the marginal contribution that he provides to the coalition of players who have entered the room before him. The imputation that is arrived at through this process is w^{π}, and it is in the core of the game.
We have so far studied two families of games in which the core is nonempty. Using the Bondareva-Shapley theorem we have proved that the core of a market game is never empty. In convex games we explicitly find

points that are in the core. Some another classes of games with a nonempty core are Spanning tree games and flow game.
Core is set-solution concept, therefore it have some limitations. As core can have many solutions it create a problem to expect a determined solution. And for some games the core is empty, what will happen in those kind of games? To overcome such issues we will move from the set-solution towards a single-valued solution concept for coalitional games, that is the *Shapley value*.

References

[MSZ] Michael Maschler, Eilon Solan, Shmuel Zamir, "Game Theory," cambridge university press