CS698A : Selected Topics in Mechanism Design **Jan-Apr 2018**

Lecture 6: Market Games

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6.1 Recap: Bondareva - Shapely Theorem

In the previous lecture, we tried to find a solution to the n-player Transferable-Utility games and when would the players form a grand-coalition. We then defined the notions of imputation, coalitional- rationality, core and balanced collection of coalitions. The lecture concluded with the characterization theorem giving the necessary and sufficient condition for having non-empty core in a TU game.

6.1.1 Bondareva - Shapely Theorem

Theorem 6.1 *The necessary and sufficient condition for a TU game (N, v) to have a non-empty core is that for every balanced coalition $\mathcal{D}$ and for every balancing weights $(\delta_S)_{S\in\mathcal{D}}$,*

$$v(N) \geq \sum_{S\in\mathcal{D}} \delta_S.v(S) \tag{6.1}$$

The theorem is useful in discarding games with empty cores and proving classes of games having non-empty cores.
Let a special set of balanced collection be $\mathcal{D}^* = 2^N$ and the corresponding balanced* weights be λ^*, where

$$\sum_{S\subseteq N, i\in S} \lambda^*(S) = 1, \forall i \in N \tag{6.2}$$

Using this, we now define an equivalent formulation of the above theorem.

Theorem 6.2 *A TU game (N,v) has a non-empty core if and only if for all balanced* weights λ^*, we have :*

$$v(N) \geq \sum_{S\subseteq N} \lambda^*(S).v(S) \tag{6.3}$$

A coalition game satisfying the Bondareva - Shapely Theorem (inequality 6.3) is called a **Balanced Game**.

6.1.2 Proof of B-S Theorem (6.2)

Consider the following Linear Program to check the feasibility of the core:

$$minimize \quad \sum_{i\in N} x_i \qquad s.t. \quad \sum_{i\in S} x_i \geq v(S) \qquad \forall S \subseteq N \tag{6.4}$$

Clearly, the solution (OPT) to this problem is at least v(N).

Claim: If there is a non-empty core then the OPT = v(N).

The optimal value is clearly at least v(N). Also, if OPT is more than v(N), then the core would be empty. Therefore, **the Core is non-empty ⇔ OPT = v(N)**

Using our knowledge of LP-Duality, we can rewrite the equation 6.4 as

$$maximize \quad \sum_{S \subseteq N} \lambda(S).v(S) \qquad s.t. \quad \sum_{S \subseteq N: i \in S} \lambda(S) = 1 \qquad \forall i \in N \quad where \quad \lambda(S) \geq 0, \forall S \in N. \tag{6.5}$$

Here, the constraints are same as that of balanced* weights. On applying weak duality:

$$\sum_{S \subseteq N} \lambda(S).v(S) \leq \sum_{i \in N} x_i^* = v(N) \tag{6.6}$$

In this equation, the inequality holds for all balanced* weights, whereas the equality between the primal solution and v(N) holds because of non-emptiness of the core.

6.2 Market Games

We now see some coalitional games, which arise naturally in practice such as Market Games. We will try to apply the B-S theorem to prove non-empty cores.

6.2.1 Framework

- **Producers:** N = { 1,2, . . . ,n }
- **Commodities:** C = { 1,2, . . . ,L }

  For Example: Raw materials like wood, metal, human resources, expert consultation hours etc.
- **Commodity Vector** is denoted by $x \subseteq \Re^L_{\geq 0}$, where x_j is the amount/quantity of commodity j, j=1,2, . . . ,L, assuming these are fluid items
- **Bundle:** $x_i \in \Re^L_{\geq 0}$ is the bundle of the producer i. x_{ij} is the amount of commodity j that producer i gets.
- **Production/Utility function:** $u_i : \Re^L_{\geq 0} \mapsto \Re$

  $u_i(x_i)$ represents the monetary value generated by producer i when given a bundle x_i.
- **Initial Endowment:** $a_i \in \Re^L_{\geq 0}$ is called the initial endowment of producer i.

6.2.2 The Coalition Strategy

If a coalition S forms, the members trade/pool the commodities among themselves. The goal is to maximize the total money (utility) generated.

Total Endowment of S: $a(S) = \sum_{i \in S} a_i$

The coalition can only redistribute these items among its members, $x_i \in \Re^L_{\geq 0}$, with $\sum_{i \in S} x_i = a(S)$.

Hence, by redistributing the items, they can generate a collective utility of $\sum_{i \in S} u_i(x_i)$.

A **Market** is defined by the vector $< N, C, (a_i, u_i)_{i \in N} >$,

where a_i and u_i respectively represent the initial endowment and production function of producer i.

The set of allocations of coalition S is defined as

$$X^S = \{(x_i)_{i \in S} : x_i \in \Re^L_{\geq 0}, \forall i \in S, x(S) = \sum_{i \in S} x_i = a(S)\}$$

Observation: X^S is compact (closed and bounded) $\forall S \subseteq N$.

Assumption: All production functions are continuous.

Worth of Coalition S:

$$v(S) = max\{\sum_{i \in S} u_i(x_i) : x = (x_i)_{i \in S} \in X^S\} \tag{6.7}$$

Since $u_i's$ are continuous and X^S is compact, the maxima is attained within the set.

6.2.3 Example

We define the market $< N, C, (a_i, u_i)_{i \in N} >$ for the example as follows:

N = { 1,2,3 } C = { 1,2 }

$a_1 = (1, 0)$ $a_2 = (0, 1)$ $a_3 = (2, 2)$

$u_1(x_1) = x_{11} + x_{12}$ $u_2(x_2) = x_{21} + 2x_{22}$ $u_3(x_3) = \sqrt{x_{31}} + \sqrt{x_{32}}$

We compute the valuations for subsets of N for the market as follows:

$v(1) = 1$ $v(2) = 2$ $v(3) = 2\sqrt{2}$

$v(1, 2) = 3$ $v(1, 3) = 5.5$ $v(2, 3) = 8.375$

We now compute the utility of the grand coalition, i.e. v(1,2,3). As we know that 2 provides at least as much utility as 1 for any commodity distribution, we can safely assume that x_1=(0,0) (i.e. all commodities of 1 are given to 2). The optimization problem now reduces to

$$v(1, 2, 3) = max\{x_{21} + 2x_{22} + \sqrt{3 - x_{21}} + \sqrt{3 - x_{22}} : 0 \leq x_{21}, x_{22} \leq 3\}$$

Solving the maximization problem independently for x_{21} and x_{22}, we get v(1,2,3) = 9.375 where

$x_1 = (0, 0)$ $x_2 = (\frac{11}{4}, \frac{47}{16})$ $x_3 = (\frac{1}{4}, \frac{1}{16})$

Definition: A Coalition Game (N,v) is a Market Game if

$$\exists L > 0, \forall i \in N \quad \exists a_i \in \Re^L_{\geq 0} \quad and \quad u_i : \Re^L_{\geq 0} \mapsto \Re$$

are continuous and concave $\forall i \in N$ s.t. the condition 6.2.2 is satisfied for all non-empty $S \subseteq N$.

6.3 Shapley - Shubik Theorem (1969)

Theorem 6.3 *The core of a market game is non-empty.*

6.3.1 Proof

We will use the B-S theorem to prove this result.

To prove: Every market game is a balanced game.

Consider a Market Game $< N, C, (a_i, u_i)_{i \in N} >$, fix an arbitrary coalition S.

Let $x^S = (x_i^S)_{i \in S}$ be the allocation that maximizes $\sum_{i \in N} u_i(x_i^S)$ - by the definition of $u_i, x^S \in X^S$, we have

- $x_i^S \in \Re^L_{\geq 0}$
- $x^S(S) = \sum_{i \in S} x_i^S = \sum_{i \in S} a_i = a(S)$
- $\sum_{i \in S} u_i(x_i^S) = v(S)$

Let $\delta = (\delta_S)_{S \subseteq N}$ be a balanced weight vector (arbitrary).

To show, $v(N) \geq \sum_{S \subseteq N} \delta_S v(S)$.

We define the allocation, $z_i = \sum_{\{S \subseteq N : i \in S\}} \delta_S x_i^S$

Claim: z_i is a feasible bundle, i.e. $\sum_{i \in N} z_i = a(N)$

Proof:

$$z(N) = \sum_{i \in N} z_i = \sum_{i \in N} \sum_{S \subseteq N : i \in S} \delta_S x_i^S$$

$$= \sum_{i \in N} \sum_{S \subseteq N} I\{i \in S\} \delta_S x_i^S = \sum_{S \subseteq N} \sum_{i \in N} I\{i \in S\} \delta_S x_i^S$$

$$= \sum_{S \subseteq N} \sum_{i \in S} \delta_S x_i^S = \sum_{S \subseteq N} \delta_S \sum_{i \in S} x_i^S$$

$$= \sum_{S \subseteq N} \delta_S x^S(\delta) = \sum_{S \subseteq N} \delta_S a(S)$$

$$= \sum_{S \subseteq N} \delta_S \sum_{i \in S} a_i = \sum_{S \subseteq N} \sum_{i \in S} \delta_S a_i$$

$$= \sum_{i \in N} \sum_{\{S \subseteq N : i \in S\}} \delta_S a_i = \sum_{i \in N} a_i \sum_{\{S \subseteq N : i \in S\}} \delta_S$$

$$= \sum_{i \in N} a_i (1) = a(N).$$

By definition of v,

$$v(N) \quad \geq \sum_{i \in N} u_i(z_i) \quad = \sum_{i \in N} u_i(\sum_{\{S \subseteq N : i \in S\}} \delta_S x_i^S)$$

By concavity, we have $f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda) f(y)$

$$\geq \sum_{i \in N} \sum_{\{S \subseteq N : i \in S\}} = \quad = \sum_{S \subseteq N} \sum_{i \in S} \delta_S u_i(x_i^S)$$

Using similar arguments as above,

$$\sum_{S \subseteq N} \delta_S \sum_{i \in S} u_i(x_i^S) \quad = \sum_{S \subseteq N} \delta_S v(S) \qquad \textit{[Balanced Condition]}$$

Result:- As this is the necessary and sufficient condition for B-S theorem, hence the Market Game has a non-empty core.

6.3.2 Restricted Market Games

Changing the Market from $< N, C, (a_i, u_i)_{i \in N} >$ to $< N, C, (a_i, u_i)_{i \in s} >$ (due to absence of some producers,

Let (S,$\tilde{v}$) be the reduced game, then $\forall T \subseteq S$,

$$\tilde{v}(T) = \quad max \quad \{\sum_{i \in T} u_i(x_i) \quad : x_i \in \Re^L_{\geq 0} \quad \forall i \in T, \quad \sum_{i \in T} x_i = \sum_{i \in N} a_i\} \quad = v(T) \tag{6.8}$$

Here, we define a restriction of v in (N,v) to the v restricted to S, which is same as v(T) $\forall T \subseteq S$. Hence we consider the subgame (S,v) of the market game (N,v).

6.3.3 Totally Balanced Games

Corollary 6.4 *If (N,v) is a Market Game, every sub-game (S,v) of it is a Market Game, and in particular is balanced.*

Such games are called **Totally Balanced Games**.

A coalition game is totally balanced if every sub-game of it has a non-empty core.

The Shapley-Shubik Theorem can also be interpreted as:

Market game is totally balanced.

It is notable that the converse is also true.

Theorem 6.5 *Every totally balanced game is a Market Game.*

6.4 Summary

In this lecture, we started with statements of the Bondareva - Shapely Theorem and their proofs. Next, we looked forward to a new class of games: 'Market Games'. After formulation, we defined the Coalition

Strategy in these games along with some of its properties and examples. We then learned the Shapley - Shubik Theorem, along with its proof which states the non-emptiness of the core of a Market Game. We finally defined totally balanced games and saw its equivalence with Market Games.