CS698A: Selected Topics in Mechanism Design **Jan-Apr 2018**

Lecture 5: Transferable Utility Games

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Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor.*

5.1 Recap from the last class

In the last lecture we discussed Nash Bargaining Theorem. We saw, with help of some examples, that the Nash Bargaining Solution did not capture the coalition characteristic reasonably. In this lecture we start with a brief introduction to the TU games and then introduce a solution concept that takes into account the possibility that players can form a coalition.

5.1.1 TU games: A brief introduction

We say utility is transferable if one player can losslessly transfer part of its utility to another player. Such transfers are possible if the players have a common currency that is valued equally by all. A Transferable Utility(TU) game is represented as (N, v) where

$$
\begin{array}{rcl}
N & : & \text{set of Players } (= \{1, 2, ..., n\}) \\
v & : & \text{characteristic function of the TU game } (N, v), v : 2^n \to \mathbb{R} \\
v(S) & : & \text{Value of the coalition } S \subseteq N \\
v(\emptyset) & = & 0
\end{array}
$$

5.1.2 Special Classes of TU Games

1. **Monotonic Games**
   $v(C) \leq v(D), \forall\ C \subseteq D \subseteq N$
2. **Superadditive Game**
   $v(C \cup D) \geq v(c) + v(D), \forall\ C, D \subseteq N$
3. **Convex Games**
   $v(C \cup D) + v(C \cap D) \geq v(C) + v(D),\ \ \forall\ C, D \subseteq N$
   Convex games $\subset$ Superadditive games

We begin this lecture with the following two questions:

- Which coalition will form? If a coalition is formed, how does it divide the worth among its members?
- What would a neutral mediator advise the players?

Answer to the former question is difficult. In course of this lecture we will assume that the grand coalition is formed (and hence try to find the condition(s) under which this happens) and try to see how players divide the total wealth among themselves in a rational way.

5.2 Some Definitions

Definition 5.1. *An **imputation** $x \in \mathbb{R}^n$ is a share of the players satisfying:*

1. $x_i \geq v(\{i\})$
2. $\sum_{i \in N} x_i = v(N)$

example:
$N = \{1, 2, 3\}$
$v(1) = v(2) = v(3) = 0$
$v(1, 2) = 2\,;\;\; v(2, 3) = 4\,;\;\; v(1, 3) = 3$
$v(1, 2, 3) = 7$

Note: Imputations are not guaranteed to exist for all games.

Definition 5.2. *An allocation $x \in \mathbb{R}^n$ is **coalitionally rational** if $\sum_{i \in S} x_i \geq v(S), \forall\;\; S \subseteq N$*

Definition 5.3. *An imputation is in the **core** of a TU game (N, v) if it is coalitionally rational i.e,*

1. $\sum_{i \in S} x_i \geq v(S), \forall\;\; S \in N$
2. $\sum_{i \in N} x_i = v(N)$

The core is collection of all coalitionally rational imputations.

Remarks:

- *The core of a coalitional game is the intersection of a finite number of half-spaces, and is therefore a convex set.*
- *The core is a compact set*

5.2.1 Computing core for various TU games

1. **Divide the Dollar(version 1):**

   $N = \{1, 2, 3\}$
   $x_1, x_2, x_3 \geq 0$
   $x_1 + x_2 \geq 0\,;\;\; x_2 + x_3 \geq 0\,;\;\; x_1 + x_3 \geq 0$
   $x_1 + x_2 + x_3 = 7$
   So for the given TU game the core is represented by the set of points
   $C(N, v) = \{(x_1, x_2, x_3) \;\mid\; x_1 + x_2 + x_3 = 7; x_1, x_2, x_3 \geq 0\}$

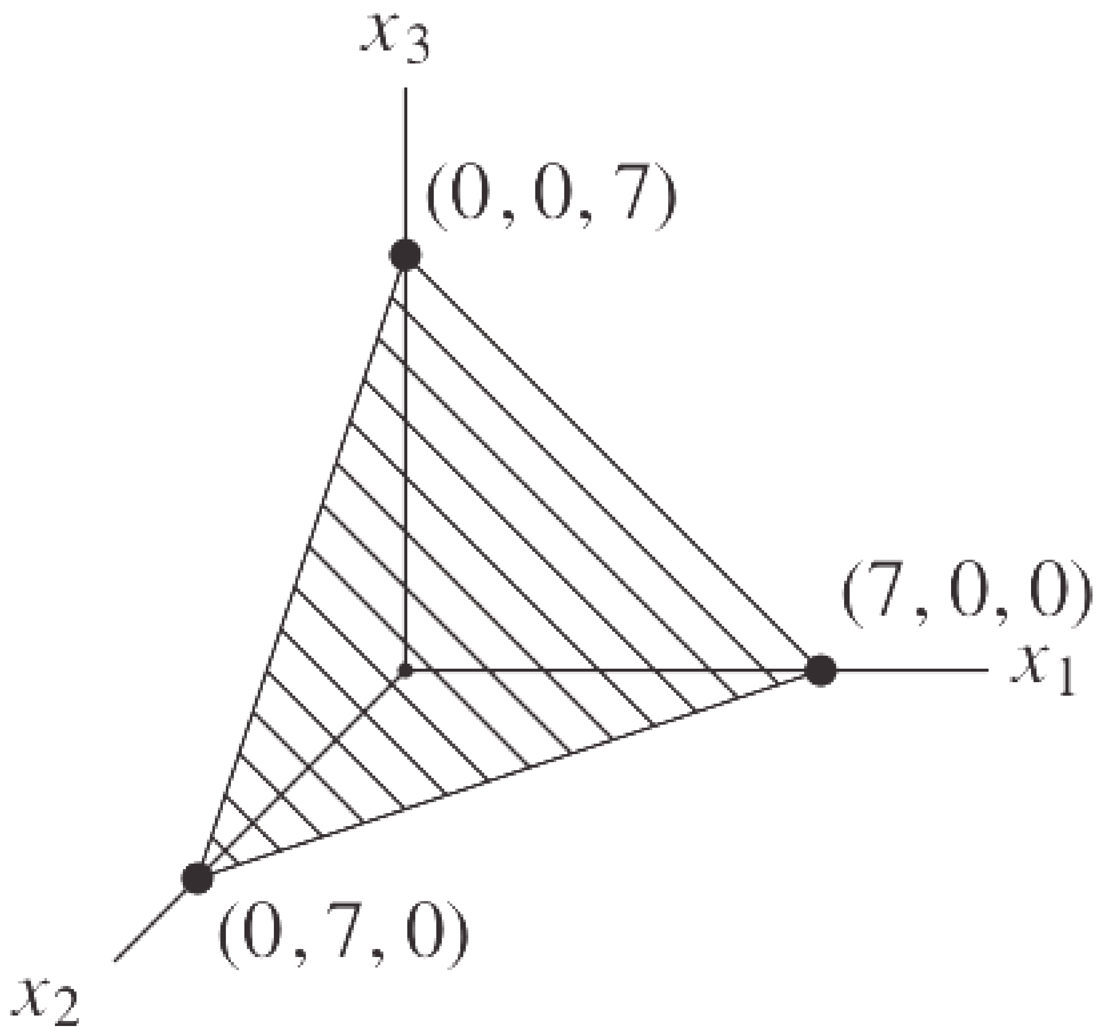

Figure 5.1: Core of Divide the Dollar(version 1)

2. **Divide the Dollar(version 2):**
   $v(1,2) = 7 \Rightarrow x_1 + x_2 = 7$;
   $x_1 + x_2 + x_3 = 7$
   From the above two conditions we get the core as
   $C(N,v) = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 7; x_3 = 0; x_1, x_2 \geq 0\}$

3. **Divide the Dollar(version 3):**
   $v(1,2) = 7 \Rightarrow x_1 + x_2 = 7$
   $v(1,3) = 7 \Rightarrow x_1 + x_3 = 7$
   $x_1 + x_2 + x_3 = 7$
   From the above three conditions we obtain that $x_2 = x_3 = 0$ and $x_1 = 7$
   So in this case core is singleton, i.e, $C(N,v) = \{(7,0,0)\}$

4. **Divide the Dollar(version 4):**
   $v(1,2) = 7 \Rightarrow x_1 + x_2 = 7$

$v(1,3) = 7 \Rightarrow x_1 + x_3 = 7$
$v(2,3) = 7 \Rightarrow x_2 + x_3 = 7$
$x_1 + x_2 + x_3 = 7$
In this case the system of equations in x_1, x_2 and x_3 is inconsistent. So, the core is empty.

The above examples raise the following question: *When is the core non-empty?*

5.3 Necessary and Sufficient condition for non-empty core in TU games

Suppose (N, v) is a coalitional game with three players, $N = \{1, 2, 3\}$. An imputation $x \in C(N, v)$ iff the following inequalities hold:

$$x_1 + x_2 + x_3 = v(1,2,3) \tag{5.1}$$

$$x_1 + x_2 \geq v(1,2); x_1 + x_3 \geq v(1,3); x_2 + x_3 \geq v(2,3) \tag{5.2}$$

$$x_1 \geq v(1); x_2 \geq v(2); x_3 \geq v(3) \tag{5.3}$$

Using above equations we get

$$v(1,2,3) \geq v(1) + v(2) + v(3) \tag{5.4}$$

$$v(1,2,3) \geq v(1,2) + v(3) \tag{5.5}$$

$$v(1,2,3) \geq v(1,3) + v(3) \tag{5.6}$$

$$v(1,2,3) \geq v(2,3) + v(1) \tag{5.7}$$

$$v(1,2,3) \geq (1/2)v(1,3) + (1/2)v(1,2) + (1/2)v(2,3) \tag{5.8}$$

If there exists a solution to the system of equations (5.4) - (5.8) then the system of equations (5.1) - (5.3) will also be consistent. Hence, this gives us the necessary and sufficient condition for non-zero core for a three player TU game.

The inequalities in Equations (5.4) - (5.8) impose the requirement that $v(N)$ be **sufficiently large**. The right-hand sides of these equations all contain collections of coalitions multiplied by various coefficients. This is summarized in the following table:

Collection of Coalitions	Coefficients
{1},{2},{3}	1,1,1
{1,2},{3}	1,1
{1,3}, {2}	1,1
{2,3}, {1}	1,1
{1,2},{1,3},{2,3}	1/2,1/2,1/2

Definition 5.4. *For every coalition S, the* ***incidence vector*** *of the coalition is the vector $\chi^S \in \mathbb{R}^N$ defined as follows:*

$$\chi_i^S = \begin{cases} 1 & \textit{if } i \in S; \\ 0 & \textit{otherwise.} \end{cases}$$

Coalition	Incidence matrix	Coefficients
$\{1\}$	1 0 0	1
$\{2\}$	0 1 0	1
$\{3\}$	0 0 1	1
$\{1,2\}$	1 1 0	1
$\{3\}$	0 0 1	1
$\{1,3\}$	1 0 1	1
$\{2\}$	0 1 0	1
$\{2,3\}$	0 1 1	1
$\{1\}$	1 0 0	1
$\{1,2\}$	1 1 0	$\frac{1}{2}$
$\{1,3\}$	1 0 1	$\frac{1}{2}$
$\{2,3\}$	0 1 1	$\frac{1}{2}$

Figure 5.1: The incidence matrices for balanced collections of coalitions in three-player games

Observe in the above table that $c^T\mathbb{I} = (1,1,1)$ for all cases where c is vector of coalitions, $\mathbb{I}$ is the incidence matrix. A set of coalitions which has positive coefficients satisfying this property is called a balanced coalition.

Definition 5.5. *A collection of coalitions $\mathscr{D}$ is a* **balanced collection** *if $\exists$ a vector of positive numbers $(\delta_S)_{S\in\mathscr{D}}$ such that*

$$\sum_{\{S\in\mathscr{D}|i\in S\}} \delta_S = 1, \qquad \forall\ i \in N$$

If $\mathscr{D}_1$ and $\mathscr{D}_2$ are two balanced coalitions such that their balancing weights are $(\delta^1_S)_{S\in\mathscr{D}_1}$ and $(\delta^2_S)_{S\in\mathscr{D}_2}$ respectively then for every $0 < \lambda < 1$, the vector of weight $(\delta^*_S)_{S\in\mathscr{D}_1\cup\mathscr{D}_2}$ will be a balancing weight for $\mathscr{D}_1\cup\mathscr{D}_2$ where $(\delta^*_S)_{S\in\mathscr{D}_1\cup\mathscr{D}_2}$ is given by the formula

$$\delta^*_S = \begin{cases} \lambda\delta^1_S & \text{if } S \in \mathscr{D}_1\backslash\mathscr{D}_2; \\ (1-\lambda)\delta^2_S & \text{if } S \in \mathscr{D}_2\backslash\mathscr{D}_1; \\ \lambda\delta^1_S + (1-\lambda)\delta^2_S & \text{if } S \in \mathscr{D}_1\cap\mathscr{D}_2. \end{cases}$$

5.3.1 Bondareva - Shapely Theorem

Statement 1: The necessary and sufficient condition for a TU game (N, v) to have a non-empty core is that for every balanced coalition $\mathscr{D}$ and for every balancing weights $(\delta_S)_{S\in\mathscr{D}}$

$$v(N) \geq \sum_{S\in\mathscr{D}} \delta_S v(S)$$

Note: The theorem holds even when the balancedness condition is relaxed to weakly balanced because the inequality holds for every balanced collection iff it holds for every weakly balanced collection.
Bondareva-Shapley theorem is a characterization result. It is useful to test (and discard) games with empty core. It is also useful in proving that certain classes of games have non-empty core.

Let a special set of balanced collection be $\mathscr{D}^* = 2^N$
and the corresponding *balanced** vector of weights be λ^*. We know that

$$\sum_{\{S\subseteq N \mid i\in S\}} \lambda^*(S) = 1, \ \ \forall \ \ i \in N$$

An equivalent statement of Bondareva-Shapley theorem in terms of above defined quantities can be given as follows

Statement 2: A TU game (N, v) has a non-empty core iff for all *balanced** weights λ^* we have

$$v(N) \geq \sum_{S\subseteq N} \lambda^*(S)v(S)$$

5.4 Summary

In this lecture, we started with the objective to find a solution to the n-player TU games which takes into account the possibility of players forming a coalition. In due course of the lecture we came across the notions of imputation, coalitional rationality, core, balanced collection of coalitions. We concluded the lecture by stating the Bondareva-Shapely Theorem which gives necessary and sufficient condition for having non-empty core.