CS698A: Selected Topics in Mechanism Design	**Jan-Apr 2018**

Lecture 4: Axiomatic Bargaining Problem and Transferable Utility Games

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4.1 Nash Bargaining solution

Theorem 4.1 *Given two-person bargaining problem (F,v), there exists a* ***unique*** *f that satisfies axioms (1-5), which is defined as*

$$f^N(F,v) = \underset{x \in F, x_1 \geq v_1, x_2 \geq v_2}{argmax} N(x,v) \tag{4.1}$$

where, N(x,v) is known as Nash product and is defined as:

$$N(x,v) = (x_1 - v_1)(x_2 - v_2)$$

The theorem is proved in two parts. The part(1), which proves the statement that the Nash bargaining solution satisfies the five axioms, has been already done in lecture 3. Let us prove the left part of theorem.

Part(2.) Given: $f(F,v)$ is a bargaining solution which satisfies axioms (1)-(5).
To show: $f(F,v) = f^N(F,v)$ where, f^N is defined in eq (4.1).

Proof: In Part(1) of the proof, it is proved that f^N satisfies all the five axioms. As, it is essential bargaining problem, therefore, $x_1^* \geq v_1$ and $x_2^* \geq v_2$ and atleast one inequality should be strict. By using the property of *Scale Covariance* we can transform the given feasible allocation set and the disagreement point (F,v) to (G,w). Let us transform F as,

$$L(x_1,x_2) = (\lambda_1 x_1 + \mu_1, \lambda_2 x_2 + \mu_2)$$

where,

$$\lambda_1 = \frac{1}{x_1^* - v_1}, \lambda_2 = \frac{1}{x_2^* - v_2}, \mu_1 = \frac{-v_1}{x_1^* - v_1}, \mu_2 = \frac{-v_2}{x_2^* - v_2}$$

Hence, the transformation is as,

$$L(x_1,x_2) = \Big(\frac{x_1 - v_1}{x_1^* - v_1}, \frac{x_2 - v_2}{x_2^* - v_2}\Big), \forall x = (x_1,x_2) \in F$$

The transformation $L(x)$ is such that the solution $f^N(f,v)$ transformed to $f^N(G,w) = (1,1)$ and the disagreement point $w = (0,0)$, the origin. Now, the objective function is

$$f^N(G,w) = \underset{x \in G, x_1 \geq w_1, x_2 \geq w_2}{argmax} N(x,w)$$

$$f^N(G,w) = \underset{x \in G, x_1 \geq w_1, x_2 \geq w_2}{argmax} (x_1 \cdot x_2)$$

Due to such a devised transformation,

$$f^N(G,w) = (1,1)$$

After transformation we have, $G = \{L(x) : x \in F\}$, $f^N(G,w) = L(x^*) = (1,1)$ and $w = L(v) = (0,0)$ as shown in figure(4.1).

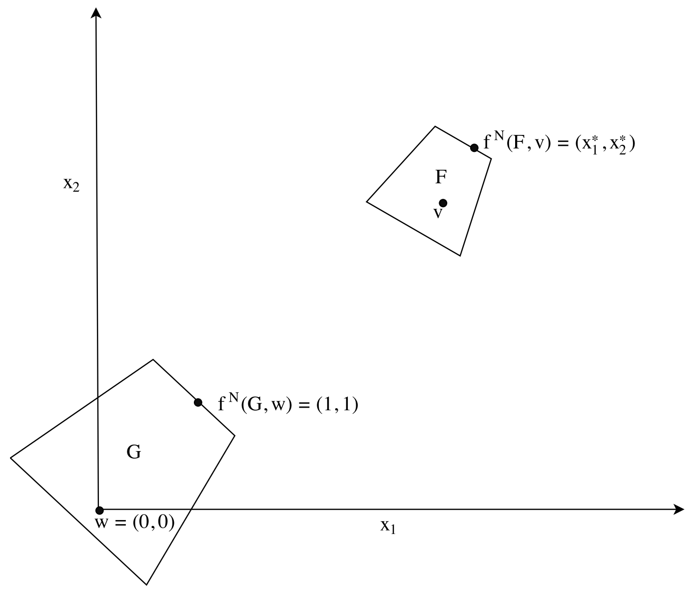

Figure 4.1: Transformation of (F, v) to (G, w)

And, $f(F,v) = f^N(f,v) \iff f(G,w) = f^N(G,w) = (1,1)$
Hence, to prove the Part(2) of Nash theorem, we need to show that, $f(G,w) = (1,1)$. As $f^N(G,w) = (1,1)$, it is clear from the definition of f^N that $(1,1)$ is the point where, the Nash product (in this case $x_1 \cdot x_2$) is maximum and hence,

$$x_1 \cdot x_2 \leq 1 \qquad \forall x \in G \tag{4.2}$$

To complete the prove we will use the claim (4.2).

Claim 4.2 $y_1 + y_2 \leq 2$, $\forall(y_1, y_2) \in G$

Proof: *Suppose the claim is incorrect and hence,* $\exists y = (y_1, y_2) \in G$ *such that* $y_1 + y_2 > 2$.
The figure(4.2) represents the whole scenario. The point y *must lie at the right side of the line* $y_1 + y_2 = 2$. *And as* G *is convex, and the points* $(1,1)$ *and* $y \in G$, *for some* $0 \leq \lambda \leq 1$, *the point* $z = \lambda y + (1-\lambda)(1,1)$ *will belong to* G.
For sufficiently small $\lambda > 0$, $z_1 \cdot z_2 > 1$. *But, from eq(4.2)* $z_1 \cdot z_2, \leq 1$, *which leads to contradiction. Therefore, it is proved that the claim is correct and* $y_1 + y_2 \leq 2$, $\forall(y_1, y_2) \in G$. ■

Enclose G with H such that H is *symmetric* w.r.t. line $x_1 = x_2$ with $(1,1)$ on the boundary of H as represented in figure (4.3).

From the property of *SPE* and *symmetry* $\Rightarrow f(H,(0,0)) = (1,1)$
From property of *IIR*, $\Rightarrow f(G,(0,0)) = f(H,(0,0)) = (1,1)$. Hence, it is proved that $f(F,v) = f^N(F,v)$ ■

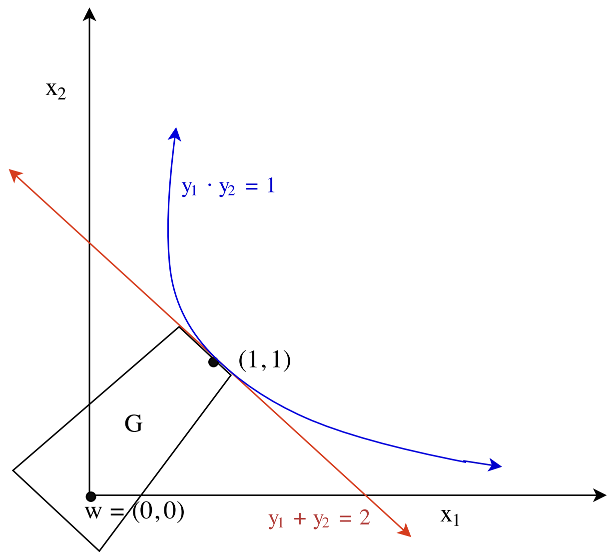

Figure 4.2: Figure for the proof of claim

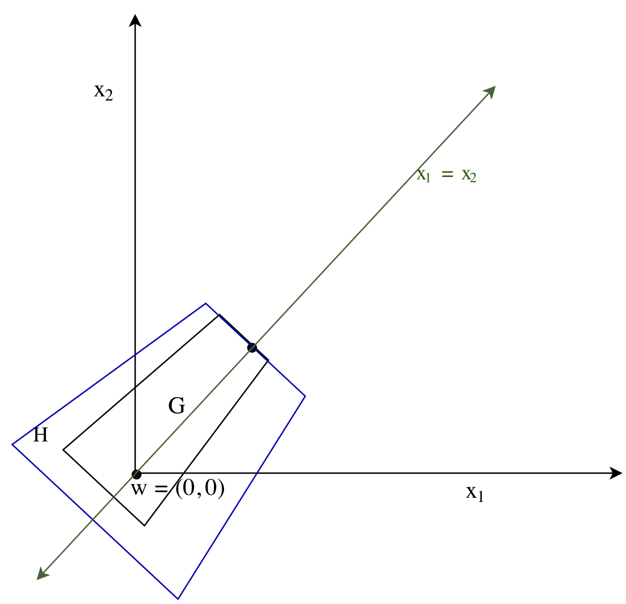

Figure 4.3: H Enclosed G and is *symmetric* w.r.t. line $x_1 = x_2$ with $(1,1)$ on the boundary of H

Here is an exercise:

Exercise 1 : Extend the proof for inessential bargaining problem.
Exercise 2 : Find atleast one other solution for any combination of three properties among SPE, symmetry, Scale covariance and IIA.

4.2 Multi-person Bargaining problem

The game in this settings is defined by $(F, (v_1, v_2, ..., v_n))$. The question arises in this context is, whether the Nash bargaining solution reasonably captures the coalitional characteristics, or not? Let us see some examples with their different versions to understand it better.

4.2.1 Example 1. Divide The Dollar:

4.2.1.1 Version 1:

There are three players and they have to divide the total wealth of amount 300 among them. Thus, formally the game is as
$N = \{1, 2, 3\}$, Total Wealth =300 and the feasible set of allocation (F) is defined as:

$$F = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 \leq 300, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}$$

In this game each player suggest a division on total wealth which can be denoted as the strategy of the players. So, the strategy of player i, $s_i = (x_1^i, x_2^i, x_3^i)$. In this version of the game the division will be implemented only if all the players agrees to that division, otherwise they all gets zero. The utility of player i, $\forall i \in N$ is as:

$$u_i(s_1, s_2, s_3) = \begin{cases} x_i, & \text{if } s_1 = s_2 = s_3 = (x_1, x_2, x_3) \\ 0 & \text{otherwise} \end{cases}$$

It is clear that each player has equal power in the game and the disagreement point is $(0, 0, 0)$. Hence, the Nash solution is $(100, 100, 100)$. This solution is perfectly reasonable as no agent or a group of agents can have profitable deviation.

4.2.1.2 Version 2:

In this version, the definition of F is same but the utility is as:

$$u_i(s_1, s_2, s_3) = \begin{cases} x_i, & \text{if } s_1 = s_2 = (x_1, x_2, x_3) \\ 0 & \text{otherwise} \end{cases}$$

The disagreement point (which is $(0, 0, 0)$) and the feasible allocation set F are same as version 1. Therefore, we can conclude that the Nash solution for the version 2 remains same as that of version 1, which is $(100, 100, 100)$. But, this is not reasonable here as the group $\{1, 2\}$ have more power than agent 3. Therefore, the group $\{1, 2\}$ may deviate to get more profit and the resultant allocation can be different from Nash solution, for example, $(150, 150, 0)$. Thus, we can conclude that the Nash solution is relevant only if no coalition other than the grand coalition(of N players) can negotiate effectively.
Effective coalition means that the members of a coalition of players can negotiate effectively and form an effective coalition if the players of that coalition change their strategies to such a feasible strategies which are beneficial to all of them.

4.2.1.3 Version 3:

In this version, the utility is defined as:

$$u_i(s_1, s_2, s_3) = \begin{cases} x_i, & \text{if } s_1 = s_2 = (x_1, x_2, x_3) \text{ or } s_1 = s_3 = (x_1, x_2, x_3) \\ 0 & \text{otherwise} \end{cases}$$

As before the Nash solution is $(100, 100, 100)$ and is definitely not reasonable as groups $\{1, 2\}$ or group $\{1, 3\}$ can deviate profitably. The agent one gets more power than other two agents. Therefore, the agent one can make very biased offers and the other players do not have any other choice but to accept it.

4.2.1.4 Version 4:

In this version, the utility is defined as:

$$u_i(s_1, s_2, s_3) = \begin{cases} x_i, & \text{if } s_1 = s_2 = (x_1, x_2, x_3) \text{ or } s_1 = s_3 = (x_1, x_2, x_3) \text{ or } s_2 = s_3 = (x_1, x_2, x_3) \\ 0 & \text{otherwise} \end{cases}$$

The Nash solution is same as before but not reasonable. The negotiation can go forever as for every proposal among a pair of agents, the other agent always has a proposal which is profitable for him and at least one of the agent of the pair.

From the four versions of the Division of Dollar game, it is clear that some better solutions and models for the coalition in the game of three or more players are needed.

4.3 Transferable Utility Games

The *transferable utility* assumption implies that there is a commodity called money that the players can freely transfer among themselves such that the payoff of any player increases one unit for every unit money he gets. With the assumption of TU, the cooperative game can be described by the *characteristic function* $v : 2^N \to R$. $v(C)$ = value of coalition $C \subseteq N$. $v(\phi) = 0$.

Definition 4.3 *A transferable Utility game is given by the tuple (N, v) where, N is set of players and v is the characteristic function.*

4.3.1 Example 1. Divide the Dollar

Let us take the example of game Divide the Dollar.
For version 1 the chacterstics function will be as: $v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = 0$ and $v(\{1, 2, 3\}) = 300$.
For version 2: $v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{1, 3\}) = v(\{2, 3\}) = 0$ and $v(\{1, 2\}) = v(\{1, 2, 3\}) = 300$.
For version 3: $v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{2, 3\}) = 0$ and $v(\{1, 2\}) = v(\{1, 3\}) = v(\{1, 2, 3\}) = 300$.
And for version 4: $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$ and $v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = v(\{1, 2, 3\}) = 300$.

4.3.2 Example 2. Minimum Cost Spanning Tree Game

Suppose multiple users are to be connected to a shared facility or resource. Each user can connect and use the facility directly or can connect to some other user who is already connected. A typical example of this game is shown in the figure with three nodes $((1), (2), (3))$ as the users and the fourth node (F) as the resource. The edge weight represents the cost to be paid to follow that edge. Suppose each user gets a benefit of 10 units if he gets connected to F directly or indirectly.

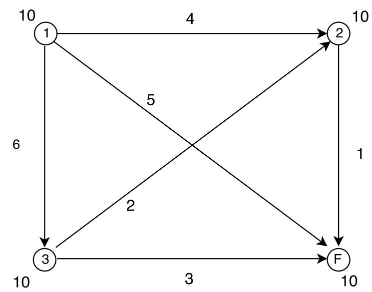

Figure 4.4: The Graph

For a given coalition $S \subseteq N$, the minimum cost to be paid by the users of coalition S is sum of the cost of edges followed by each of them to connect with the facility F, which is the cost of the minimum cost spanning tree connects the users who belongs to that coalition S. The characteristic function will be as:
$v(\{1\})$ = benefit_of(1) - cost $= 10 - 5 = 5$
$v(\{2\}) = 10 - 1 = 9$
$v(\{3\}) = 10 - 3 = 7$
$v(\{1,2\}) = (10 + 10) - (4 + 1) = 15$
$v(\{1,3\}) = (10 + 10) - (5 + 3) = 12$
$v(\{2,3\}) = (10 + 10) - (1 + 2) = 17$
$v(\{1,2,3\}) = (10 + 10 + 10) - (4 + 1 + 2) = 23$

4.3.3 Example 3. Bankruptcy Game(E, c)

Suppose a bank has the assets of money E, $E \geq 0$ and can think of as the value the bank will get if the asset is sold. c is a vector of claims made by the agents. The value of coalition $S \subseteq N$ is as:

$$v(S) = \Big[E - \sum_{i \in N \setminus S} c_i\Big]^+$$

where,

$$x^+ = max\{x, 0\}$$

A typical example is $N = \{1, 2, 3\}$, $c = (10, 50, 70), E = 100$.
The characteristic function is:
$v(\{1\}) = 0, v(\{2\}) = 20, v(\{3\}) = 40, v(\{1,2\}) = 30, v(\{1,3\}) = 50, v(\{2,3\}) = 90, v(\{1,2,3\}) = 100$

4.3.4 Special Classes of TU Games:

4.3.4.1 Monotonic Game:

A TU game (N, v) is a monotonic game if $v(C) \leq v(D)$, $\forall C \subseteq D \subseteq N$.

4.3.4.2 Superadditive Games:

A TU game (N, v) is superadditive if $v(C \cup D) \geq v(C) + v(D)\ \forall C, D \subseteq N, C \cap D = \phi$. Monotonic and Superadditive are independent features.

Exercise 3 : Find out all four possible combinations of MONO and SA.

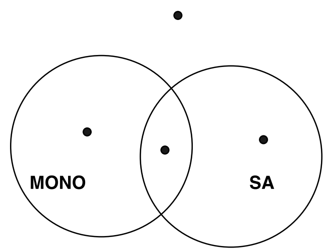

Figure 4.5: all four possible combinations.

4.3.4.3 Convex Games:

A TU game (N, v) is a convex game if $v(C \cup D) + v(C \cap D) \geq v(C) + v(D)$, $\forall C, D \subseteq N$.

Note: Suppose C and D are disjoint sets, $C \cap D = \phi$.
Then we obtain, $v(C \cup D) \geq v(C) + v(D)\ \forall C, D \subseteq N, C \cap D = \phi$.
Thus, every convex game is superadditive but the converse need not be true.

Proposition: (N, v) is a convex game iff
$v(C \cup \{i\}) - v(C) \leq v(D \cup \{i\}) - v(D) \qquad \forall C \subseteq D \subseteq N \setminus \{i\}$.

Thus, the convex game is a game in which the marginal contribution of a player to a coalition is smaller than or equal to his marginal contribution in any superset of the coalition.

References

[YN] Narahari Y., "Game Theory And Mechanism Design," *IISC lecture notes series*