**CS698A: Selected Topics in Mechanism Design** **Jan-Apr 2018**

# Lecture 3: Axiomatic Bargaining Problem

*Lecturer: Swaprava Nath* *Scribe(s): Piyush Bagad*

**Disclaimer**: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor.*

## 3.1 Recap: Axiomatic Bargaining

At the end of the last class, we introduced the two-agent bargaining problem which involves two agents negotiating on a mutually beneficial arguement which is self-enforcing. In this lecture, we will look at desirable properties that bargaining solutions must possess and also state a solution concept by Nash.

### 3.1.1 Setup and Notations

For the two-agent bargaining problem, we are given a tuple $\langle F, v \rangle$ described as follows:

- **Allocation set:** $F \subseteq \mathbb{R}^2$ denotes the *feasible set of allocations*
- **Disagreement point:** $v = (v_1, v_2) \in \mathbb{R}^2$ denotes the *disagreement point i.e.* in case all the negotiations fail, then agent $i$ gets $v_i$ amount of share, for $i = 1, 2$.
- **Assumptions:** We make the following assumptions that can be justified based on the nature of problem in consideration.
  1. $F \subseteq \mathbb{R}^2$ is convex and closed
  2. $F \cap \{(x_1, x_2) \mid \; x_i \geq v_i, i = 1, 2\} \neq \phi$

#### 3.1.1.1 Illustration of a feasible set and Disagreement point

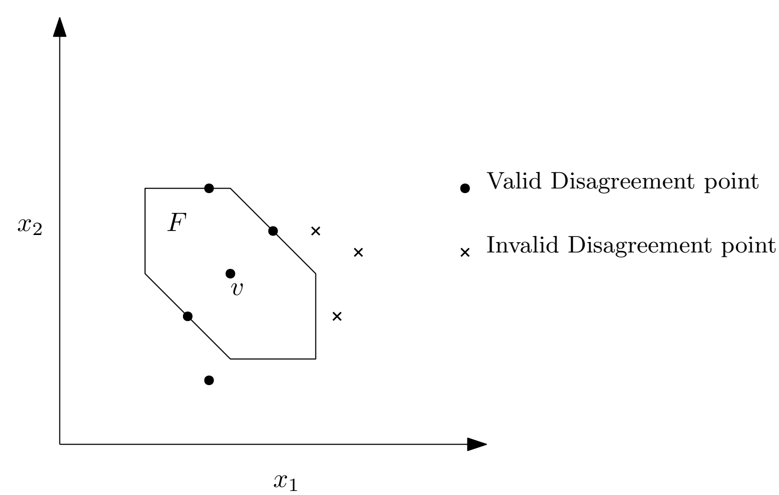


Figure 3.1: Vaild and invalid Disagreement points

## 3.2 Axioms: Desirable properties of Bargaining solutions

The mechanism designer would want to find an allocation $f(F, v) = (f_1(F, v), f_2(F, v)) \in F$. However, in such a setup, the question we will address is the following: what are the desirable properties we need the solution function $f$ to satisfy?

### 3.2.1 Axiom 1: Strong (Pareto) Efficiency

Given a feasible set of allocations $F$, we say that $x = (x_1, x_2) \in F$ is *strongly (Pareto) efficient* if $\nexists$ another $y = (y_1, y_2) \in F$ s.t. $y_i \geq x_i, \forall i = 1, 2$ and the inequality being strict for at least one agent.
On similar lines, we can define **Weak (Pareto) Efficiency** as follows:
Given a feasible set of allocations $F$, we say that $x = (x_1, x_2) \in F$ is *weakly (Pareto) efficient* if $\nexists$ another $y = (y_1, y_2) \in F$ s.t. $y_i > x_i, \forall i = 1, 2$.
It is easy to observe that SPE $\Rightarrow$ WPE but not the other way round.

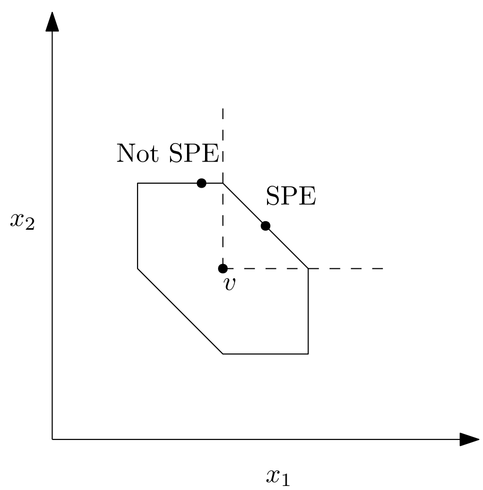


Figure 3.2: Illustrative example

### 3.2.2 Axiom 2: Individual Rationality

IR states that each player should get an allocation greater than that obtained on a disagreement point during the bargaining. More formally,

$$f(F, v) \geq v \Rightarrow f_i(F, v) \geq v_i, \forall i = 1, 2 \tag{3.1}$$

### 3.2.3 Axiom 3: Scale Covariance

Consider an affine transformation of the feasible space $F$, *i.e.* let $\lambda_1, \lambda_2 > 0, \mu_1, \mu_2 \in \mathbb{R}$ and define

$$G := \{(\lambda_1 x_1 + \mu_1, \lambda_2 x_2 + \mu_2) \mid (x_1, x_2) \in F\} \tag{3.2}$$

$$w := (\lambda_1 v_1 + \mu_1, \lambda_2 v_2 + \mu_2) \tag{3.3}$$

Basically, we are scaling and translating the feasible allocation space through the affine transformation. A solution function $f$ is said to satify *Scale Covariance* if for such a transformation, $(\lambda_1 f_1(F,v)+\mu_1, \lambda_2 f_2(F,v)+\mu_2)$ is a solution for $(G,w)$.

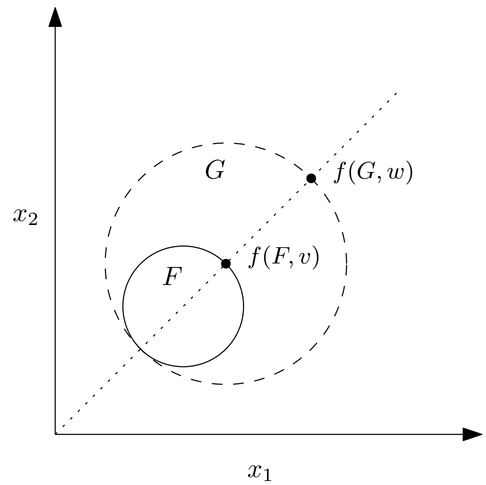


Figure 3.3: Illustrative example

### 3.2.4 Axiom 4: Independence of Irrelevant alternatives

For any closed and convex set $F$, the solution function satisfies IIA if

$$G \subseteq F, f(F,v) \in G \Rightarrow f(G,v) = f(F,v) \tag{3.4}$$

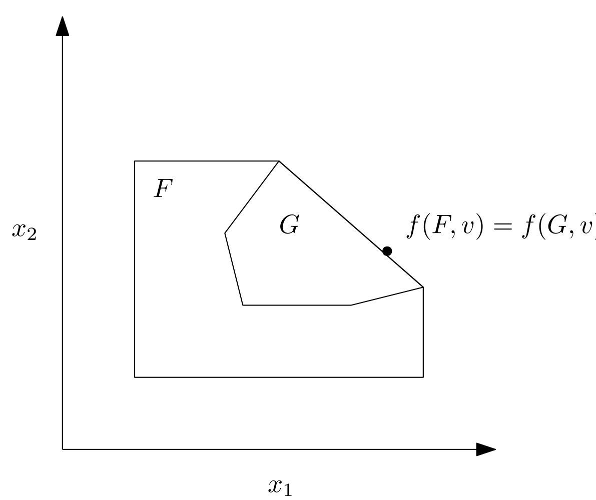


Figure 3.4: Illustrative example

### 3.2.5 Axiom 5: Symmetry

If positions of agents are symmetric, then the solution function also must treat them symmetrically. More formally, Symmetry axiom can be stated as

$$v_1 = v_2, \{(x_2, x_1) \mid (x_1, x_2) \in F\} \subseteq F \Rightarrow f_1(F, v) = f_2(F, v) \tag{3.5}$$

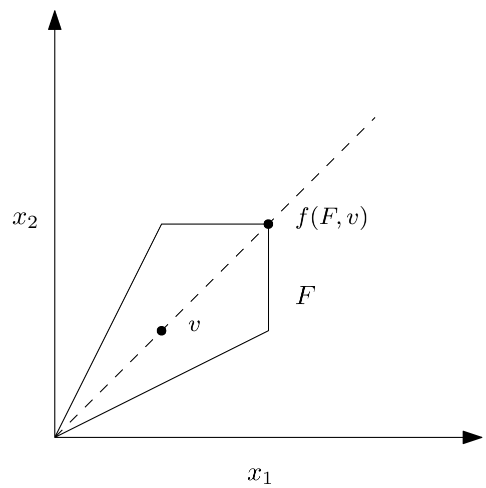


Figure 3.5: Illustrative example

## 3.3 The Nash Bargaining Solution

**Theorem 3.1** *Given a two-person bargaining problem $(F, v)$, there exists a unique solution function $f$ that satisfies the axioms 1 to 5 and is given by:*

$$f(F, v) \in \underset{(x_1,x_2)\in F,\quad x_1 \geq v_1, x_2 \geq v_2}{\text{argmax}} ((x_1 - v_1)(x_2 - v_2)) \tag{3.6}$$

*The product $N(x, v)$ is called the Nash product. Further, we will show that the solution function is unique and the maximum value of the Nash product is taken on a unique point.*

### 3.3.1 Illustrative example

Let the set of allocations be $F$ = Convex Hull of $(0, 4), (1, 1), (4, 0)$, and the disagreement point be $v = (1, 1)$ The allocation $f(F, v) = (2, 2)$ is Strong Pareto Efficient, satisfies Individual Rationality and Symmetry axiom. To observe Scale Covariance (see figure 3.8), consider:

Obtain $G$ using $\lambda_1 = \lambda_2 = 1/2, \mu_1 = \mu_2 = 1 \Rightarrow f(G, w) = (2, 2) \Rightarrow$ satisfies SC for this transformation

Obtain $H$ using $\lambda_1 = \lambda_2 = 1/2, \mu_1 = \mu_2 = 0 \Rightarrow f(H, u) = (1, 1) \Rightarrow$ satisfies SC for this transformation

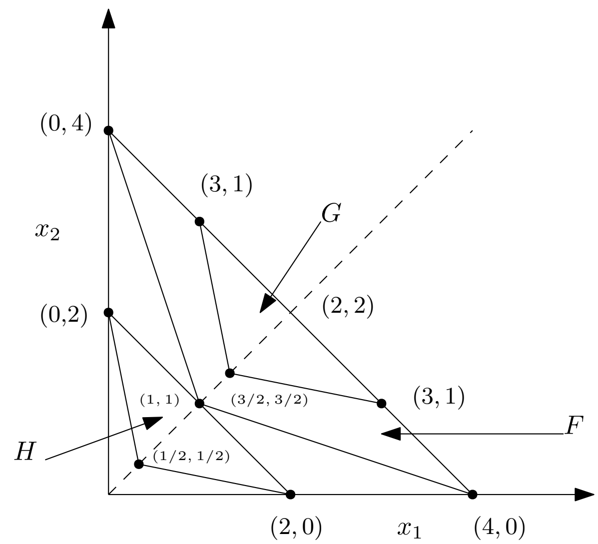


Figure 3.6: Illustration of Nash bargaining problem

**Note: (Essential Bargaining Problem)** We will consider a special (but almost general) subclass of the problem where $\exists$ at least one $y = (y_1, y_2) \in F$ s.t. $y_1 > v_1$ and $y_2 > v_2$. We call this subproblem as the *Essential Bargaining Problem.* For the most general case that includes the boundaries of $F$ also, we will derive the solution later.

**Definition 3.2** *A function defined over a convex, non empty set $S$ denoted by $g : S \to \mathbb{R}$ is said to be quasi-concave if*

$$g(\lambda x + (1-\lambda)y) \geq min\{g(x), g(y)\} \quad \forall x, y \in S, \forall \lambda \in [0, 1] \tag{3.7}$$

*Similarly, $g$ is said to be strictly quasi-concave if*

$$g(\lambda x + (1-\lambda)y) > min\{g(x), g(y)\} \quad \forall x, y \in S, \forall \lambda \in [0, 1] \tag{3.8}$$

**Definition 3.3** *Alternative definition of (strict) quasi-concavity: If $g$ is (strict) quasi-concave then the Upper Contour Set of $f$ defined as follows*

$$U(f, a) = \{x \in S \ \mid \ g(x) \geq a\} \tag{3.9}$$

*is (strictly) convex $\forall a \in \mathbb{R}$.*

**Example of a Quasi-concave function:** In the illustration in figure 3.7, the functions $f, g$ are quasi-concave.

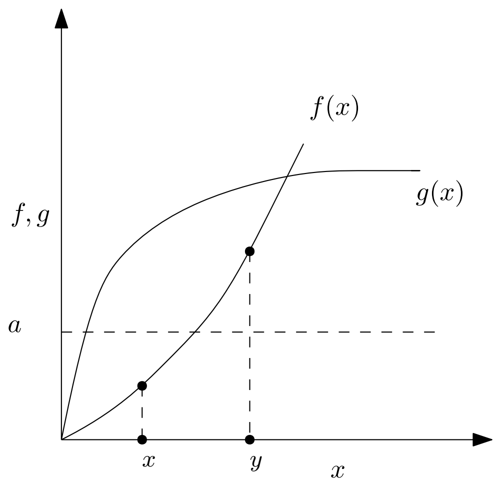


Figure 3.7: Quasi-concave functions: examples

**Observation:** The Nash product $N(x, v)$ is *strictly quasi-concave* for the Essential Bargaining problem in the region $x_1 \geq v_1, x_2 \geq v_2$.

**Fact:** We will assume the following result as a fact without having to prove it: A strict *quasi-concave* function has a unique maxima.
Thus, note that the solution to the Essential bargaining problem will be unique.

The Nash Bargaining solution states that the axioms 1 to 5 are satisfied for the unique bargaining solution

$$f^N(F, v) = \underset{(x_1,x_2)\in F,\quad x_1\geq v_1, x_2\geq v_2}{\text{argmax}} (N(x)) \tag{3.10}$$

$N(x)$ is called the Nash product.

**Proof:**

- **(Part 1)** Claim: $f^N$ satisfies axioms 1 to 5. Let $f^N(F, v) = (x_1^*, x_2^*)$

  1. **(Strong Efficiency)** We have

     $$x^* = (x_1^*, x_2^*) = \underset{(x_1,x_2)\in F,\quad x_1\geq v_1, x_2\geq v_2}{\text{argmax}} (N(x)) \tag{3.11}$$

     Suppose $x^*$ is not SE. Then there exists $y = (y_1, y_2)$ s.t. $y_1 \geq x_1^*$, $y_2 \geq x_2^*$ and at least one of them is strict. But, for an essential bargaining problem, $N(x^*) > 0$ and by assumption, $N(y) > N(x^*) > 0$. But this contradicts the definition of $x^*$. Thus, $x^*$ is Strongly efficient solution.
  2. **(Individual Rationality)** IR follows directly from the definition of $x^*$.
  3. **(Scale Covariance)** Consider $\lambda_1, \lambda_2 > 0, \mu_1, \mu_2 \in \mathbb{R}$ and define

     $$G = \{(\lambda_1 x_1 + \mu_1, \lambda_1 x_1 + \mu_1) \mid (x_1, x_2) \in F\} \tag{3.12}$$

     The Nash product problem in $G$ will be:

     $$\max_{y\in G, y_i\geq v_i} (y_1 - w_1)(y_2 - w_2) \tag{3.13}$$

where $w_i = \lambda_i v_i + \mu_i, i = 1, 2$. Thus the maximization objective will become

$$\Rightarrow \max_{x \in F, x_i \geq v_i} \lambda_1 \lambda_2 (x_1 - v_1)(x_2 - v_2) \tag{3.14}$$

The maximum is attained at $x^* = (x_1^*, x_2^*)$. Hence, the solution to the problem in $G$ is given by $y^* = (\lambda_1 x_1^* + \mu_1, \lambda_2 x_2^* + \mu_2)$. Thus, $f^N$ satisfies Scale Covariance.

4. **(Independence of Irrelevant Alternatives)** Let $G \subseteq F$ be convex and closed. We have $x^* = (x_1^*, x_2^*)$ optimal in $(F, v)$ and let $y^* = (y_1^*, y_2^*)$ be optimal in $(G, w)$, and let $x^* \in G$. Since $G \subseteq F$, $N(x^*) \geq N(y^*)$. But since $y^*$ is optimal in $G$, $N(y^*) \geq N(x^*)$. Thus, $N(x^*) = N(y^*)$. But since the maxima point is unique (as observed in section 3.3), we will have $x^* = y^*$.
5. **(Symmetry)** Suppose $F$ is symmetric set *i.e.* $F = \{(x_2, x_1) \mid (x_1, x_2) \in F\}$ and $v_1 = v_2 = v$ By definiton, $x^*$ maximizes $N(x_1^*, x_2^*) = (x_1^* - v_1)(x_2^* - v_2) = N(x_2^*, x_1^*)$. However, since the optima is unique, we have $x_1^* = x_2^*$.

- **(Part 2)** Suppose $f(F, v)$ is a bargaining solution satisfying axioms 1 to 5.
To show that: $f(F, v) = f^N(F, v)$, where $f^N$ is as defined earlier in Theorem 3.1. We will only present a rough sketch of the proof in this lecture. The plan is to tranform $(F, v)$ to $(G, w)$ such that the solution to the transformed problem is given by $(1, 1)$ and point $w = (0, 0)$. Formally, taking help of Scale Covariance,

$$f(F, v) = f^N(F, v) \Leftrightarrow f(G, (0, 0)) = f^N(G, (0, 0)) = (1, 1) \tag{3.15}$$

Finally, we will need to show $f(G, (0, 0)) = (1, 1)$. The following illustration can help better understand the proof sketch.

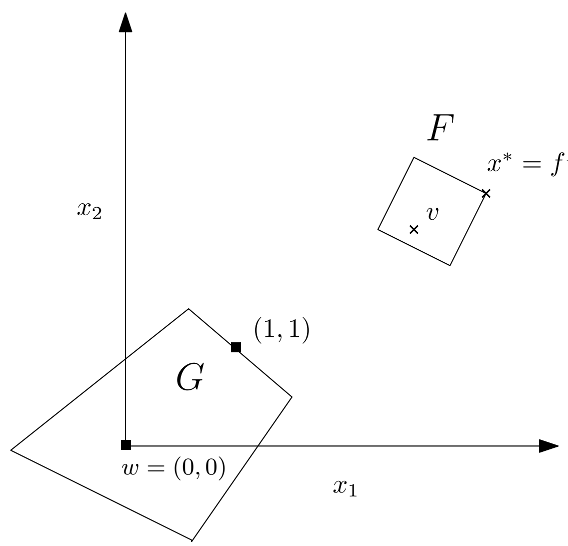


Figure 3.8: Illustration of the proof sketch

■

## 3.4 Summary

In this lecture, we looked to find solutions to the two-agent bargaining problem through an axiomatic approach. We noted the desirable properties or axioms a general solution function must have and concluded that the only possible solution satisfying all the axioms will be the Nash bargaining solution.