CS698A: Selected Areas of Mechanism Design

Midterm – Semester 2, 2017-18.
Computer Science and Engineering
Indian Institute of Technology Kanpur
Total Points: 40, Time: 2 hours
ATTEMPT ALL QUESTIONS

The following definition can be useful.

Definition 1 *A coalitional game* (N, v) *is a* **weighted majority game** *if there exists a quota* $q \geqslant 0$ *and nonnegative real weights* $(w_i)_{i \in N}$*, one for each player, such that the value of each nonempty coalition* S *is*

$$v(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} w_i \geqslant q, \\ 0 & \text{if } \sum_{i \in S} w_i < q. \end{cases}$$

This is a *simple* game, where a coalition can ensure positive value if their collective weight reaches the threshold of the quota.

Q. (1) Consider a weighted majority game (N, v), with *four* players, i.e., $N = \{1, 2, 3, 4\}$, weights $w_1 = 6, w_2 = 1, w_3 = 2, w_4 = 4$, and quota to be $q = 7$.

(a) Write down the valuations of each coalition for this game.

(b) Find the core of this game.

(c) Find the Shapley value of this game.

3 + 3 + 4 points.

Q. (2) Let $(\{1, 2, 3\}, v)$ be a cooperative game defined as follows: $v(\emptyset) = 0, v(1) = v(2) = 1, v(3) = v(1, 2) = 2, v(1, 3) = v(2, 3) = 4, v(1, 2, 3) = 5$.

(a) Find its core.

(b) Compute the Shapley value for this game.

(c) Is the game convex?

4 + 4 + 2 points.

Q. (3) Let (N, v) be a coalitional game satisfying the following **strong symmetry property**: for every permutation π over the set of players, and for every coalition $S \subseteq N$,

$$v(S) = v(\pi(S)), \qquad \text{where } \pi(S) := \{\pi(i) : i \in S\}.$$

Prove the following claims.

(a) The core of this game is nonempty if and only if for every coalition $S \subseteq N$,

$$v(S) \leqslant \frac{|S|}{n} v(N).$$

(b) If the core is nonempty, and there exists a coalition $\emptyset \neq S \subset N$, satisfying $v(S) = \frac{|S|}{n} v(N)$, then the core is a singleton with the only imputation

$$\left(\frac{v(N)}{n}, \ldots, \frac{v(N)}{n} \right).$$

5 + 5 points.

Q. (4) Compute the Shapley values of the players in the weighted majority game with $n+1$ players, with weight $\frac{n}{3}$ for player 1 and weight 1 for every other n players. The quota is given by $q = \frac{n}{2}$. Assume n is divisible by 6. What is the limit of the Shapley values of the players as $n \to \infty$?

(4 + 4) + 2 points.

Good Luck!