A Proof Outline for Q(3)

First notice that the given condition translates to every S having same cardinality to have same valuation, hence we can write

$$v(S) = v_k, \quad \forall S \text{ s.t. } |S| = k \quad \forall k = 1, \dots, n.$$

(a) Core nonempty

$\Rightarrow$ $\exists$ x s.t. x is an imputation and

$$\sum_{i \in S} x_i \geq v_k \qquad \forall S \text{ s.t. } |S| = k, \forall k = 1, \dots, n$$

For each k, if we sum all these inequalities, note every x_i will appear exactly $\binom{n-1}{|S|-1}$ times on the left and there are $\binom{n}{|S|}$ such inequalities, hence

$$\binom{n-1}{|S|-1} \underbrace{\sum_{i \in N} x_i}_{= v(N)} \geq \binom{n}{|S|} v(S)$$

$$\Rightarrow \frac{|S|}{n} v(N) \geq v(S)$$

The reverse direction is immediate for the imputation $\left(\frac{v(N)}{n}, \dots, \frac{v(N)}{n}\right)$ □

(b) Clearly, the given imputation is in core. Suppose $\exists$ another x (different from the one given) also in core

Let S_0 be a nonempty subset s.t.

$$v(S_0) = \frac{|S_0|}{n} v(N)$$

clearly, by the previous argument, $v(T) = v(S_0)$ $\forall T$ s.t. $|T| = |S_0|$.

since x is in core

$$\sum_{i \in T} x_i \geq v(S_0) = \frac{|S_0|}{n} v(N) \qquad \forall T \text{ s.t. } |T| = |S_0|$$

$$\Rightarrow \sum_{i \in N \setminus T} x_i \leq v(N) \left(\frac{n - |S_0|}{n} \right) \qquad \cdots (1)$$

Consider a permutation π of the players s.t.

$$\underbrace{x_{\pi(1)} \leq x_{\pi(2)} \leq \cdots}\ \leq x_{\pi(n)}$$

Consider first $\tilde{T}$ s.t. $|\tilde{T}| = |S_0|$

as x is in core $\quad \sum_{i \in \tilde{T}} x_i \geq v(\tilde{T}) = v(S_0) = \frac{|S_0|}{n} v(N)$

Then $\exists$ at least one $i \in \tilde{T}$ s.t. $x_i \geq \frac{v(N)}{n}$

hence $\forall j \in N \setminus \tilde{T}$ $\quad x_j \geq x_i \geq \frac{v(N)}{n}$

but by (1) $\quad \sum_{j \in N \setminus \tilde{T}} x_j \leq (n - |S_0|) \frac{v(N)}{n}$

together all $x_j = \frac{v(N)}{n}$ $\quad \forall j \in N \setminus \tilde{T}$

and hence $\forall i \in \tilde{T}$ $x_i = \frac{v(N)}{n}$.

Hence only the given imputation is in core $\quad \square$