A Simple Proof of Bernoulli's Inequality

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Bernoulli's inequality states that for $r \geq 1$ and $x \geq -1$:

$$(1+x)^r \geq 1+rx$$

The inequality reverses for $r \leq 1$.

In this note an elementary proof of this inequality for rational r is described. The proof is only based on the fact that for any n non-negative numbers, geometric mean can not exceed arithmetic mean[1, Section 250,253]. I think, part(s) of this proof may be appearing as exercise(s) in some text book(s), but unfortunately as search engines are unable to find those, I have written this note.

Let us first consider the case when $r \leq 1$, i.e., let $r = \frac{p}{q}$ with $p \leq q$.

Geometric mean of q numbers ($(1+x)$ occurs p times):

$$1, 1, \ldots, 1, (1+x), (1+x), \ldots (1+x)$$

is $(1+x)^{p/q}$ and arithmetic mean of these numbers is: $1 + \frac{p}{q}x$. As geometric mean is less than arithmetic mean, we get

$$(1+x)^{p/q} \quad \leq \quad 1 + \frac{p}{q}x \tag{1}$$

Or substituting $r = \frac{p}{q}$,

$$(1+x)^r \leq 1+rx$$

This proves the inequality for $r \leq 1$ case.

Next consider the case when exponent is more than 1. Condition (1) is equivalent to:

$$(1+x) \leq \left(1 + \frac{p}{q}x\right)^{q/p}$$

Let $y = \frac{p}{q}x$, as $x \geq -1$, $y \geq -\frac{p}{q} \geq -1$.

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As $x = \frac{q}{p}y$, in terms of y we have:

$$\left(1 + \frac{q}{p}y\right) \quad \leq \quad (1+y)^{q/p} \tag{2}$$

Now $s = \frac{q}{p} \geq 1$ is an arbitrary rational number. Hence, for any rational number $s \geq 1$ and $y \geq -1$, we have

$$(1+y)^s \geq 1 + sy$$

This proves the inequality for $s \geq 1$ case.

As these inequalitiess are true for all rational numbers $r \leq 1$ and $s \geq 1$, they are also true for all real numbers. This is because, any real number can be approximated by rational numbers to arbitrary precision (this formally follows from Cauchy construction of real numbers).

References

[1] H.S.Hall and S.R.Knight, Higher Algebra, 1887.