Algorithms on Graphs with Small Dominating Targets

Divesh Aggarwal, Chandan K. Dubey, and Shashank K. Mehta **

Indian Institute of Technology, Kanpur - 208016, India,
{divesh,cdubey,skmehta}@iitk.ac.in

Abstract. A dominating target of a graph $G = (V, E)$ is a set of vertices T s.t. for all $W \subseteq V$, if $T \subseteq W$ and induced subgraph on W is connected, then W is a dominating set of G. The size of the smallest dominating target is called dominating target number of the graph, $dt(G)$. We provide polynomial time algorithms for *minimum connected dominating set, Steiner set*, and *Steiner connected dominating set* in dominating-pair graphs (i.e., $dt(G) = 2$). We also give approximation algorithm for *minimum connected dominating set* with performance ratio 2 on graphs with small dominating targets. This is a significant improvement on $appx \leq d(opt + 2)$ given by Fomin et.al. [1] on graphs with small d-octopus.

Classification: Dominating target, d-octopus, Dominating set, Dominating-pair graph, Steiner tree.

1 Introduction

Let $G = (V, E)$ be a simple (no loops, no multiple edges) undirected graph. For a subset $Y \subseteq V$, $G(Y)$ will denote the induced subgraph of G on vertex set Y i.e. $G(Y) = (Y, \{(x, y) \in E : x, y \in Y\})$. Since we will only deal with *induced* subgraphs in this paper, some times only Y may be used to denote $G(Y)$. For a vertex $x \in V$, *open neighborhood* denoted by $N(x)$ is given by $\{y \in V : (x, y) \in E\}$. The *closed neighborhood* is defined by $N[x] = N(x) \cup \{x\}$. Similarly, the closed and the open neighborhoods of a set $S \subset V$ are defined by $N[S] = \cup_{x \in S} N[x]$ and $N(S) = N[S] - S$ respectively. A vertex set S_1 is said to *dominate* another set S_2 if $S_2 \subseteq N[S_1]$. If $N[S_1] = V$, then S_1 is said to dominate G.

We address four closely related domination and connectivity problems on undirected graphs; minimum connected dominating set (MCDS), Steiner connected dominating set (SCDS), Steiner set (SS), and Steiner tree (ST), each is known to be NP-complete [2]. Steiner set problem finds application in VLSI routing [3], wire length estimation [4], and network routing [5]. Minimum connected

** Partly supported by Ministry of Human Resource Development, Government of India under grant no. MHRD/CD/2003/0320

dominating set and Steiner connected dominating set problems have recently received attention due to their applications in wireless routing in ad hoc networks [6].

Many interesting graph classes such as permutation graphs, interval graphs, AT-free graphs [7, 8, 9, 10] have a pair of vertices with a property that any path connecting them is a dominating set for the graph. This pair is called a *dominating pair* of the graph. The concept of *Dominating target* was introduced by Kloks et. al. [11] as a generalization of the dominating pair. Any vertex set T in a graph $G = (V, E)$ is said to be a dominating target of G if the following property is satisfied: for every $W \subseteq V$, if $G(W)$ is connected and $T \subseteq W$, then W dominates V. The cardinality of the smallest dominating target is called the *dominating target number* of the graph G and it is denoted by $dt(G)$. The family of graphs with $dt(G) = 2$ are known as dominating-pair (DP) graphs and their dominating target is referred as dominating-pair. Minimum connected dominating set and Steiner set problems are polynomially solvable on the family of AT-free graphs [12], which is a subclass of DP. We will present here efficient algorithms for MCDS, SS, and SCDS on dominating-pair graphs.

A relevant parameter to the current work is d-octopus, considered by by Fomin et. al. [1]. A d-octopus of a graph is a subgraph $T = (W, F)$ of G s.t. W is a dominating set of G, and T is the union of d (not necessarily disjoint) shortest paths of G that have one endpoint in common. It is conjectured that $dt(G) \leq d$, where the graph has a d-octopus, [1]. Let *opt* be the optimal solution of MCDS problem and *appx* be its approximation due to the algorithm by Fomin et.al., then $appx \leq d(opt + 2)$. The complexity of this algorithm is $O(|V|^{3d+3})$. We will present an $O(|V|^{dt(G)+1})$ approximation algorithm for MCDS with performance ratio 2, which is an improvement both in terms of complexity (assuming the conjecture) and approximation factor (for an introduction on approximation algorithms see [13, 14]).

2 Problem definitions

In this paper we discuss the problem of computing following.

Minimum Connected Dominating Set (MCDS) Given a graph $G = (V, E)$, vertex set C is a *connected dominating set* (CDS) if $V = N[C]$ and $G(C)$ is connected. MCDS is a smallest cardinality CDS.

Steiner Connected Dominating Set (SCDS) Given a graph $G = (V, E)$ and a set $R \subseteq V$ of required vertices, vertex set C is a *connected* $|R|$*-dominating set* (R-CDS) if $R \subseteq N[C]$ and $G(C)$ is connected. SCDS of R is a smallest cardinality R-CDS.

Steiner Set (SS) Given a graph $G = (V, E)$ and a set $R \subseteq V$ of required vertices, vertex set S is an R*-connecting set* (R-CS) if $G(S \cup R)$ is connected. SS of R is a smallest cardinality R-CS.

Steiner Tree (ST) Given an edged-weighted graph $G = (V, E, w)$ (w is the edge-weight function) and a set $R \subseteq V$ of required vertices, a tree T is an

R-spanning tree (*R*-SPN) if it contains all *R*-vertices. ST of *R* is a minimum weight (sum of the weights of the edges) *R*-SPN.

Note that Steiner set problem is equivalent to Steiner tree problem when the edge weights are taken to be 1; and MCDS is an instance of SCDS when R is the entire V.

3 Exact algorithms on dominating pair graphs

3.1 Minimum connected dominating set

Let (u, v) be a dominating pair of the graph $G = (V, E)$ and $X = N[u]$ and $Y = N[v]$. For each $x \in X$ define $A_x = \{a : (a, x) \in E \text{ and } \{a, x\} \text{ dominates } X\}$. Define B_y in a similar way for each $y \in Y$. Now let Γ be as follows. Here $x \in X$, $y \in Y$, and $\alpha \dots \beta$ denote a shortest path between α and β.

$$\Gamma = \{P|P = u \dots v, \text{ or } u \dots by, \text{ for } b \in B_y \text{ or}$$
$$xa \dots v, \text{ for } a \in A_x \text{ or } xa \dots by, \text{ for } a \in A_y \text{ and } b \in B_y\}$$

Balakrishnan et. al. [12] have given $O(|V|^3)$ algorithms to compute MCDS and SS in AT-free graphs. They claim that the smallest cardinality path in Γ is a MCDS of the graph. Although the authors address the problem of MCDS in AT-free graphs, they do not use any property of this class other than the existence of a dominating pair. Contrary to our expectation, the algorithm does not work on all dominating pair (DP) graphs. In the graph of Figure 1 $\{x_1, x_2, x_5, x_6\}$ is an MCDS but no MCDS of size 4 is computable by their algorithm (no CDS of size 4 is in Γ).

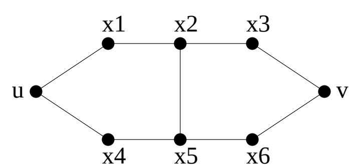

Fig. 1. A DP graph where Balakrishnan et.al. algorithm fails

Theorem 1. *Let $G = (V, E)$ be a dominating pair graph and $\{u, v\}$ any dominating pair with distance greater than 4. Then the shortest paths in Γ are MCDS of G.*

Proof. We show that if S is an MCDS then it can be transformed into another MCDS S' which belongs to Γ.

Case 1: $u \in S, v \in S$. In this case S must be a shortest path connecting u and v, which is already in Γ.

Case 2: $u \in S, v \notin S$ or $u \notin S, v \in S$. We consider the first situation only. There must exist a $y \in S \cap N(v)$. As S is connected, let P be a path from u to y contained in S. If $|S| - |P| \geq 1$ then $S' = P \cup \{v\}$ is the required MCDS in Γ. So, assume that $S = P$. Let b be the vertex in P connected to y. If $b \in B_y$ then we are done. Else there must exist a $y' \in Y$ not dominated by $\{b, y\}$. As S is a

MCDS, there must exist a $b' \in P$ s.t. $(b', y') \in E$. Then $S' = S \cup \{y', v\} - \{b, y\}$ is the required path in Γ.

Case 3: $u \notin S$, $v \notin S$. Therefore there exist S-vertices x and y such that $x \in X$ and $y \in Y$. Since S is connected there exists a path from x to y in S, say P. $P \cup \{u, v\}$ is a path connecting u and v so it must dominate entire graph. Therefore P must dominate $V - X - Y$. Further, the condition $d(u, v) > 4$ ensures that vertices that dominate any part of X are mutually exclusive from the vertices dominating any part of Y. We consider three cases.

$|S| - |P| \geq 2$ Here $S' = P \cup \{u, v\}$ is obviously in Γ.

$|S| - |P| = 1$ Let $S - P = \{p\}$. Now p must dominate either parts of X or parts of Y but not both. Without loss of generality assume that p dominates parts of X. So P must be dominating $V - X$. Thus $S' = S \cup \{u\} - \{p\}$, which is obviously connected, dominates entire V and $|S'| = |S|$. From Case 2 we know that there is a path $Q \in \Gamma$ such that it dominates V and $|Q| = |S'| = |S|$.

$|S| = |P|$ If the vertex a adjacent to x in P is in A_x and the vertex b adjacent to y in P is in B_y, then P is in Γ.

Next assume that vertex a adjacent to x in P is not in A_x or b adjacent to y in P is not in A_y. Without loss of generality assume the former. Then there must exist $x' \in X$ which is not dominated by $\{a, x\}$. Since both a and x dominate parts of X, they do not dominate any part of Y. Thus $P - \{x, a\}$ dominates Y. Let $S' = P \cup \{u, x'\} - \{x, a\}$. Clearly $S' \cup \{v\}$ is connected so it must dominate V. But $P - \{x, a\}$ dominates V so S' also dominates entire V. From Case 2 we know that there is a path $Q \in \Gamma$ such that it dominates V and $|Q| = |S'|$. But by construction $|S'| = |S|$ so $|Q| = |S|$. □

If $d(u, v) > 4$ then compute Γ and output the smallest path. In case $d(u, v) \leq 4$, then either a shortest path connecting u to v will be an MCDS or there exists an MCDS of size at most 4. This leads to an $O(|V|^5)$ algorithm to calculate an MCDS in DP graphs.

3.2 Steiner Set

Let $G = (V, E)$ be a graph and R a subset of its vertices. Define an edge-weighted graph $G_w(V, E, w)$ where $w(e) = 1$ if both vertices of the edge e are in $V - R$; $1/2$ if one vertex is in $V - R$; 0 if neither is in $V - R$. Define a function L over the paths of G as follows. Let P be a path of G and $length(P)$ denotes its length in G_w, then $L(P) = length(P) + 1$ if both end vertices of P are in $V - R$; $length(P) + 1/2$ is one end vertex of P is in $V - R$; $length(P)$ if neither end-vertex is in $V - R$. Observe that $L(P)$ is the number of $V - R$-vertices in P.

In describing the algorithm to compute Steiner set for a required set R in a dominating-pair graph, we will first assume that R is an independent set (no two R-vertices are adjacent). The general case will be shown to reduce into this case in linear time.

Theorem 2. *Let $G = (V, E)$ be a dominating-pair graph and R be an independent set of vertices in it. Then there exists a pair of vertices $u, v \in V$ such that for every minimum-L path P between u and v, $P - R$ is a Steiner set of R in G.*

Proof. Let S be a Steiner set for R in G. First we will assume that $|S| > 3$. The case of $|S| \leq 3$ will be handled by simple search. Let u', v' be a dominating pair of G. Let $P_1 = u'...u''u''' \equiv P_1'u''u'''$ be a G-shortest path from u' to the connected set $S \cup R$. Similarly let $P_2 = v'...v''v''' \equiv P_2'v''v'''$ be a G-shortest path from v' to $S \cup R$. Then u''', v''' are in $S \cup R$; $P_1 - \{u'''\}$ and $P_2 - \{v'''\}$ are outside $S \cup R$; and no vertex of P_1' or of P_2' dominates any R vertex. Observe that every path X connecting u'' and v'' dominates entire R because $P_1'.X.P_2'$ dominates entire graph. Let $u'''x_1x_2...x_{k-1}x_kv'''$ be a shortest path in $G(S \cup R)$. From the above observation $u''u'''x_1...x_kv'''v''$ dominates all the R vertices. For the convenience we will also label u''' and v''' with x_0 and x_{k+1} respectively.

Suppose there is an S-vertex s not in $\{x_i\}_{i\in[k+1]}$. Since a Steiner set is minimum, it must be dominating some R vertex which is not dominated by any x_i. Thus it must be dominated by u'' or v''. Let S' be the set of S-vertices outside $\{x_i\}_{i\in[k+1]}$. Define $S_1 = \{s \in S' : N[s] \cap R \cap N[u''] \neq \emptyset\}$ and $S_2 = \{s \in S' : N[s] \cap R \cap N[v''] \neq \emptyset\}$. From the above observation $S_1 \cup S_2 = S'$. We will show that $S_1 \cap S_2 = \emptyset$. Assume otherwise. Let $s \in S'$ such that $r_1 \in N[u''] \cap R \cap N[s]$ and $r_2 \in N[v''] \cap R \cap N[s]$. So $u''r_1sr_2v''$ is a path. From the earlier observation it dominates entire R. Thus $\{u'', s, v''\}$ is a Steiner set, but it contradicts an earlier assumption that SS has more than 3 vertices.

All paths connecting u'' to v'' dominate all R-vertices and minimum-L paths among them have L value at most $S - |S'| + 2$ because $L(u''x_0x_1...x_{k+1}v'') = |S| - |S'| + 2$. Using the path $P_3'' = u''x_0x_1...x_{k+1}v''$ we will find a pair of vertices u, v such that all paths connecting these vertices dominate R and among them minimum-L paths have $|S|$ non-R-vertices. We achieve this in two steps First we modify the u''-end of P_3'' and find u. Then work on the other end.

Case 1 $S_1 = \emptyset$. Starting from x_0, let x_{i_0} be the first S-vertex on the path $x_0, x_1, ..x_{k+1}$.

Claim Either $N[u''] \cap R \subseteq N[i_0] \cap R$ or there is an index $j > i_0$ such that $u''rx_j...x_{k+1}v''$ is a path which dominates all R-vertices and $L(u''rx_jx_{j+1}...x_{k+1}v'') \leq L(x_0x_1...x_{k+1}v'')$, where r is an R vertex.

Proof of the claim suppose u'' dominates an R vertex r which is not dominated by x_{i_0}. At least one S vertex must dominate it so let it be x_j. Consider the path $u'...u''rx_j...x_{k+1}v''$
$...v'$. It dominates the graph so the subpath $u''rx_j...x_{k+1}v''$ must dominate all R-vertices. Further the number of non-R-vertices in this path cannot exceed that of $x_0...x_{k+1}v''$ because while the former has only one new vertex, it does not have x_{i_0}, an S vertex, which is present in the latter. **end-proof**

Let $u = x_{i_0}$ if $N[u''] \cap R \subseteq N[x_{i_0}] \cap R$ else define $u = u''$. Let P_3' be the path $x_{i_0}x_{i_0+1}...x_{k+1}v''$ in the former case and $u''rx_jx_{j+1}...x_{k+1}v''$ in the latter case. Observe that in either case P_3' dominates all R-vertices (in former case there is at most one R-vertex between u'' and x_{i_0} and R-vertices do not dominate other

R-vertices) and the number of non-R-vertices on it are no more than those in $x_0...x_{k+1}v''$, which is $|S| - |S_2| + 1$.

In addition, every path connecting u to v'' must dominate all R-vertices as the following reasoning shows. The case of $u = u''$ is already established. In case $u = x_{i_0}$, pad the path at the left with $P_1'u''x_0...x_{i_0-1}$ and to the right with P_2'. This path dominates the graph. P_2' does not dominate any R-vertex and $P_1'u''x_0...x_{i_0-1}$ does not dominate any R-vertex which is not already dominated by x_{i_0}. Since one path between u and v'', namely P_3', has L value $|S| - |S_2| + 1$, the minimum-L paths between these vertices have at most $|S| - |S_2| + 1$ non-R-vertices.

Case 2 $S_1 \neq \emptyset$. Then $P_3' = u''x_0...x_{k+1}v''$ has at most $|S| - |S_2| + 1$ non-R-vertices. Define $u = u''$. All path between u and v'' dominate entire R, because $u = u''$. The minimum L paths among them cannot have more than $|S| - |S_2| + 1$ non-R vertices since $L(P_3') = |S| - |S_2| + 1$.

Together these cases imply that there exists a vertex u such that all path between u and v'' dominate entire R and the minimum-L path among them have L value at most $|S| - |S_2| + 1$.

This completes the computation of u. To determine v we repeat the argument from the other end. Let x_{j_0} be the first S vertex on the path $x_{k+1}x_k...$ starting from x_{k+1}. Then $v = v''$ if S_2 is non-empty or if $N[v''] \cap R$ is not contained in $N[x_{j_0}] \cap R$. Otherwise $v = x_{j_0}$. Repeating the argument given above we see that all paths between u and v dominate all R-vertices and there is at least one path between these vertices with at most $|S|$ non-R-vertices. Therefore we conclude that all minimum-L path between u and v have at most $|S|$ non-R-vertices. □

The algorithm to compute the Steiner set is as follows.

Data: A DP graph $G = (V, E)$ and a set $R \subseteq V$.
Result: A Steiner set for R.

1 For each set of at most 3 vertices check if it forms an R-connecting set. If any such set is found, then output the smallest of these sets;

2 Otherwise compute all-pair shortest paths on G_w. Compute the set Γ as the collection of those G_w-shortest paths that dominate R. Select a path P from Γ with minimum L-value. Output $P - R$.

Algorithm 1: Steiner set algorithm for independent set R in DP graphs

The time complexity of the first step is $O(|V|^3.(|E| + |V|))$. The cost of the second step is $O(|V|^3 + |V|^2.|E|)$ Hence the overall complexity is $|V|^3(|E| + |V|)$.

This completes the discussion for independent R case. The general case is easily reduced to this case. Let $G = (V, E)$ be a dominating pair graph and R be the required set of vertices. Shrink each connected components of $G(R)$ into a vertex. Then the resulting graph G' is also a dominating pair graph (if u, v is a dominating pair of G and u and v merge into u' and v' respectively after shrinking, then u', v' is a dominating pair of G'). Also the new required vertex set R' is an independent set in G' and each Steiner set for R' in G' is a Steiner set of R in G and its converse is also true.

3.3 Steiner connected dominating set

Definition 1. *Let G be a graph and R be a subset of its vertices. A subset of vertices D_R is called* R-dominating target *if every connected subgraph of G containing D_R dominates R. In addition, if each vertex of D_R has some R vertex in its closed neighborhood, then we call it an* essential-R-dominating-target.

Lemma 1. *For any R there exists essential R-dominating target with cardinality at most $dt(G)$.*

Proof. We present a constructive proof. Let $D = \{d_i : i \in I\}$ be a dominating target of G of size $dt(G)$. Let r_0 be any vertex in R and $\mathfrak{p}_i$ be a path from r_0 to d_i for each $d_i \in D$. Let d'_i is the first vertex from d_i on $\mathfrak{p}_i$ such that $N[d'_i] \cap R \neq \emptyset$. Let $\mathfrak{p}'_i$ is the sub-path of $\mathfrak{p}_i$ from d_i to the vertex prior to d'_i. Now we show that $D_R = \{d'_i : i \in I\}$ is an essential R dominating target. By construction, each vertex of D_R has at least one R vertex in its neighborhood. Now consider arbitrary connected set C containing D_R. Append the paths $\mathfrak{p}'_i$ to C. The resulting graph is connected and contains all vertices of D so it dominates entire G. But $\mathfrak{p}'_i$ do not dominate any R-vertices so C must be dominating all the R-vertices. □

If G is a dominating pair graph, then an essential R dominating target D_R exists with at most 2 vertices. If it is a singleton, then SCDS problem becomes trivial because this vertex dominates the entire R. So in the remainder of this section we assume that $D_R = \{u, v\}$ and denote the distance $d(u, v)$ by d_0. D_R being an essential R-dominating target, $N[u] \cap R \neq \emptyset$ and $N[v] \cap R \neq \emptyset$.

Lemma 2. *Let S be a connected set of vertices in G, i.e., the induced graph on S is connected. Then S is a connected dominating set of R iff S dominates $N_2[u] \cap R$ and $N_2[v] \cap R$, here $N_2[.]$ denotes 2-distance closed neighborhood.*

Proof. "Only if" part is trivial since $N_2[u] \cap R$ and $N_2[v] \cap R$ are subsets of R.

As $\{u, v\}$ is an essential dominating target, $N[u] \cap R$ and $N[v] \cap R$ are non-empty. Let $r_1 \in N[u] \cap R$ and $r_2 \in N[v] \cap R$. So there must be some $x \in N_2[u] \cap S$ and $y \in N_2[v] \cap S$ s.t. r_1 and r_2 are adjacent to x and y respectively. Let $S_1 = \{r_1, u\}$ and $S_2 = \{r_2, v\}$. Then $S' = S \cup S_1 \cup S_2$ is connected and contains u and v. By the definition of R-dominating target, S' dominates all R-vertices. Thus S must dominate $R - (N_2[u] \cup N_2[v])$. Combining this with the given fact that S dominates $N_2[u] \cap R$ and $N_2[v] \cap R$, we conclude that S dominates entire R. □

Lemma 3. *Let $d(u, v) \geq 5$ and S be a connected set of vertices in G containing u. If S also contains a vertex x such that $d(x, v) \leq 2$, then S dominates $N_2[u] \cap R$.*

Proof. Let Q be a shortest path from x to v. Define $S' = S \cup Q$. By construction S' is connected and contains $\{u, v\}$ therefore it dominates R. In particular, it dominates $N_2[u] \cap R$. Vertices of $Q - \{x\}$ are contained in $N[v]$ and $d(u, v)$ is at least 5, so vertices of $Q - \{x\}$ do not dominate $N_2[u] \cap R$. Therefore S must dominate $N_2[u] \cap R$. □

Lemma 4. *Let $d(u, v) \geq 5$ and S be a connected R-dominating set. Let y be a cut vertex of $G(S)$ and $G(S - \{y\})$ has a component C such that $C \cup \{y\}$ contains all the S vertices within 3-neighborhood of v. If P is a path in G connecting y and u, then $S' = C \cup P$ is also a connected R-dominating-set.*

Proof. From Lemma 2 it is sufficient to show that S' is connected and it dominates $N_2[v] \cap R$ and $N_2[u] \cap R$. Firstly, $C \cup \{y\}$ is connected so S' is also connected. Next, S is an R-dominating-set and $S \cap N_3[v]$ is contained in $C \cup \{y\}$ so $C \cup \{y\}$ dominates $N_2[v] \cap R$. Finally, $N[v] \cap R$ is non-empty and S is an R-dominating set so S contains a vertex x such that $d(x, v) \leq 2$. All S-vertices within 3-neighborhood of v are in $C \cup \{y\}$ so $x \in S'$. Further, u also belongs to S' since it is in P. Using Lemma 3 we deduce that S' dominates $N_2[u] \cap R$. This completes the proof. □

Let S be a SCDS for R. We partition it into *levels* as follows. $x \in S$ is defined to be in level i if $d(u, x) = i$. Observe that there is at least one S-vertex at level 2 and at least one S-vertex at level $d_0 - 2$. Further, if $x \in S$ is the only vertex at level i where $2 < i < d_0 - 2$, then x is a cut vertex of $G(S)$.

Lemma 5. *Let $d_0 \geq 9$. Then there exists an SCDS for R which has a unique vertex x_0 with $d(u, x_0) = d_1$ for some $d_1 \in \{3, 4\}$ and a unique vertex y_0 with $d(v, y_0) = d_2$ for some $d_2 \in \{3, 4\}$.*

We omit the proof to save the space.

Theorem 3. *Suppose G has an essential R dominating target $\{u, v\}$ with $d(u, v) \geq 9$. Then every minimum vertex set, S, among the sets satisfying the following conditions is a SCDS of R.*

(a) $G(S)$ is connected.
(b) $\exists x_0 \in S$ with $d(u, x_0) = 3$ or 4 such that x_0 is a cut vertex of $G(S)$ and a component of $G(S - \{x_0\})$, C_u, is such that $C_u \cup \{x_0\}$ dominates $N_2[u] \cap R$.
(c) $\exists y_0 \in S$ with $d(v, y_0) = 3$ or 4 such that y_0 is a cut vertex of $G(S)$ and a component of $G(S - \{y_0\})$, C_v, is such that $C_v \cup \{y_0\}$ dominates $N_2[v] \cap R$.
(d) $S - C_u - C_v$ is a shortest path between x_0 and y_0.

Proof. From Lemma 2 every set satisfying the conditions is a connected R-dominating set. Therefore if a SCDS belongs to this collection of sets, then every smallest set satisfying the conditions must be a SCDS.

From Lemma 5 there exists a SCDS, S, of R with cut vertices x_0 at distance 3 or 4 from u such that $C_u = \{x \in S : d(u, x) < d(u, x_0)\}$ is a component of $G(S - \{x_0\})$. S being an SCDS, $\{x_0\} \cup C_u$ must dominate $N_2[u] \cap R$. Similarly y_0 at a distance 3 or 4 from v in S such that condition (c) is also satisfied. If we replace $S - C_u - C_v$ by a G-shortest path between x_0 and y_0 then also the set will be a CDS, from Lemma 2. Therefore minimality of S requires that $S - C_u - C_v$ is a shortest path connecting x_0 and y_0. Therefore S is one of the CDS that satisfy the conditions. Therefore the smallest sets that satisfy the conditions must be SCDS. □

Corollary 1. *If S is an SCDS, then $|C_u| \leq d(u, x_0)$ and $|C_v| \leq d(v, y_0)$.*

Proof. If C_u is replaced by a shortest path P between u and x_0 in S, then from Lemma 4 the resulting set is also R-CDS. Besides, the optimality of S requires that $|S| \leq |S| - |C_u| + |P| = |S| - |C_u| - d(u, x_0)$. □

Algorithm 2 computes SCDS of any vertex set R in a DP graph with essential dominating pair $\{u, v\}$ with $d(u, v) \geq 9$.

Data: A DP graph $G = (V, E)$, a subset of vertices R, essential R-dominating-pair $\{u, v\}$ with $d(u, v) \geq 9$
Result: A Steiner connected dominating set of R

1 Compute all pair shortest paths;
2 **for** *all* $x \in V$ *s.t.* $d(u, x) =$ *3 or 4* **do**
3     $\mathcal{A}_x = \{P_{ux}\} \cup \{A : G(A)$ is connected, $x \in A, |A| \leq d(u, x), N_2[u] \cap R \subset N[A]\}$;
    /* P_{ux} is a shortest path between u and x */
4     $A_x =$ smallest cardinality set in $\mathcal{A}_x$;
5 **end**
6 **for** *all* $y \in V$ *s.t.* $d(v, y) =$ *3 or 4* **do**
7     $\mathcal{A}_y = \{P_{vy}\} \cup \{A : G(A)$ is connected, $y \in A, |A| \leq d(v, y), N_2[v] \cap R \subset N[A]\}$;
    /* P_{vy} is a shortest path between v and y */
8     $A_y =$ smallest cardinality set in $\mathcal{A}_y$;
9 **end**
10 $\mathcal{S} = \{A_x \cup A_y \cup P_{xy} : d(u, x) = 3$ or 4, $d(v, y) = 3$ or 4, $P_{x,y}$ a shortest path between x and $y\}$;
11 **return** *the smallest set in* $\mathcal{S}$;

Algorithm 2: SCDS algorithm for DP graphs

The correctness of the Algorithm 2 is immediate from Theorem 3. Step 1 costs $O(|V|(|V| + |E|))$. Steps 2 and 6 each costs $O(|V|^4.|R|)$ Cost of the tenth step is $O(|V|^2)$. The total complexity of the algorithm is $O(|V|^4.|R|)$.

For the case with $d_0 \leq 8$ either the SCDS is a shortest path connecting u and v or it contains at most d_0 vertices. Therefore a simple way to handle this case is to test every set of up to d_0 cardinality for connectivity and R domination and select the smallest. If no such set exists, then the shortest path is the solution. This approach costs $O(|V|^8.|R|)$. The cost of computing an essential R-dominating-target is $O(|V| + |E|)$. Adding all the costs we have following theorem.

Theorem 4. *In a dominating-pair graph the Steiner connected dominating set for any subset R can be computed in $O(|V|^8.|R|)$ time. If the distance between the R-dominating pair vertices is greater than 8, then complexity improves to $O(|V|^4.|R|)$.*

4 Approximation algorithms

Following result is by Fomin et.al.

Theorem 5 **([1]).** *Let $T = (W, F)$ be a d-octopus of a graph $G = (V, E)$, then*

- *T can be computed in $O(|V|^{3d+3})$.*
- *If $\gamma(G)$ is a minimum connected dominating set, then $|W| \le d.(\gamma(G) + 2)$.*

It is conjectured that $dt(G) \le d$ for a graph having a d octopus [1]. We will present a *appx* $\le 2\gamma(G)$ algorithm with complexity $O(|V||E| + |V|^{dt(G)+1})$. Following theorem is stated without proof.

Theorem 6. *Let $G = (V, E, w)$ be an edge-weighted (non-negative weights) graph and $R \subseteq V$ be an arbitrary set of required vertices. Then a Steiner tree of R can be calculated in $O(|V|(|V| + |E|) + (|V| - |R|)^{|R|-2}|R|^2)$.*

Corollary 2. *Let $G = (V, E)$ be a graph and $R \subseteq V$ be an arbitrary set of required vertices. Then a Steiner set for R can be computed in $O(|V|(|V| + |E|) + (|V| - |R|)^{|R|-2}|R|^2)$.*

For convenience we define $f(k) = |V|(|V| + |E|) + |V|^k(k + 2)^2$.

4.1 Computation of a minimum dominating target

Let $G = (V, E)$ be a graph. Then $T \subset V$ is a dominating target iff for all $W \subseteq V$ if $T \subseteq W$ and $G(W)$ is connected, then $N[W] = V$. The problem of computing a minimum dominating target is known to be NP-complete, [15]. Here we generalize the algorithm given in [12] to compute a dominating pair in AT-free graphs, to one that computes a dominating target in general graphs.

Lemma 6. *A set $S \subseteq V$ is a dominating target of G if and only if for every vertex $v \in V$, S doesn't lie in a single component of $G(V - N[v])$.*

First compute all neighborhood deleted components of the graph, which costs $O(|V|^{2.83})$ [16]. Starting with $t = 1$. Select each set of size t and check if it is completely contained in any of the pre-computed components. If any set is found which is not contained in any component, then it is a dominating target, otherwise increment t and repeat till one dominating target is found. This computation costs $O(dt(G) \cdot |V|^{dt(G)+1})$ time.

4.2 Minimum connected dominating set

Theorem 7. *Let $G = (V, E)$ be a connected graph with dominating target number $dt(G)$. If the cardinality of MCDS is $opt(G)$, then in $O(|V|.|E| + |V|^{dt(G)+1})$ time a connected dominating set of G can be computed with cardinality no greater than $opt(G) + dt(G)$.*

Proof. Let D be a minimum dominating target of the graph. It can be computed in $O(|V|^{dt(G)+1})$ as described in section 2.3. Let T be a Steiner tree for the required set D. Hence from the definition of dominating targets, T is a connected dominating set for G. This can be calculated by algorithm of Theorem 6 in $O(f(dt(G)-2))$.

Let M be any MCDS of G. In particular, it dominates D so $M \cup D$ is a connected set containing D. As T is the minimum connected set containing D, $|T| \leq |M \cup D| \leq |M| + |D| = |M| + dt(G)$. □

It is easy to see that $dt(G) \leq opt(G)$. So $appx \leq 2.opt(G)$.

4.3 Steiner Connected Dominating Set

Theorem 8. *Let $G = (V, E)$ be a connected graph with dominating target number $dt(G)$ and $R \subseteq V$. Let the Steiner connected dominating set (SCDS) of R have cardinality $opt(G, R)$. Then a connected R-dominating set (an approximation to SCDS for R), can be computed in $O(|V|.|E| + |V|^{dt(G)+1})$ time with cardinality no greater than $opt(G, R) + 2dt(G)$.*

Proof. As described in the proof of Lemma 1, compute an essential R-dominating-target D_R in $O(|V|^{dt(G)+1})$ time.

Compute Steiner tree of D_R, T using algorithm of Theorem 6. T is a connected set containing D_R so it dominates R. As $|D_R| \leq dt(G)$, the cost of the computation is bounded by $f(dt(G)-2)$. Next we show that $|T| \leq opt(G, R) + 2.dt(G)$.

Let S be an SCDS of R in G. D_R is an essential dominating target for R so each member of D_R is adjacent to some R vertex. For each $d \in D_R$ let r_d denote any one vertex from R which adjacent to d. Let R_D denote the set $\{r_d : D \in D_R\}$. Since S dominates R, $S \cup R_D$ is connected. Further, by construction $S \cup R_D \cup D_R$ is connected also connected. By the definition of Steiner trees T is the smallest connected set containing D_R. So $|T| \leq |S \cup R_D \cup D_R| \leq |S| + |R_D| + |D_R| \leq opt(G, R) + 2.dt(G)$. The last inequality is due to the fact that $|R_D| \leq |D_R| \leq dt(G)$. □

$opt(G, R)$ = size of the smallest connected R-dominating set $\geq$ size of the smallest R-dominating target = D_R. Therefore from the last two lines of the above proof $appx \leq 3.opt(G, R)$.

4.4 Steiner Set

Corollary 3. *Let $G = (V, E)$ be a connected graph with dominating target number $dt(G)$ and $R \subseteq V$. Let $opt(G, R)$ denote the cardinality of a Steiner set of R, then an R-connecting set (Steiner set approximation) can be computed in $O(|V|.|E| + |V|^{dt(G)+1})$ time with cardinality not exceeding $opt(G, R) + 2dt(G)$.*

Proof (sketch). Reduce G to G' by shrinking each connected component, R_i, of R to a vertex r_i. Set R' is independent in G'. Observe that if S is an R-connecting set in G, then $S \cup R'$ is the union of R' and a connected R'-dominating set in G'. Conversely if C is a connected R' dominating set in G', then $C - R'$ is a connecting set of R' is G' which is also a connecting set of R in G. Therefore we can compute a Steiner set of R by first computing SCDS of R' in G'. The claim follows from the theorem. □

Future Work: It remains to decide whether MCDS, SS, and SCDS are NP-hard on graphs with bounded dominating targets.

References

[1] Fomin, F.V., Kratsch, D., Müller, H.: Algorithms for graphs with small octopus. Discrete Applied Mathematics **134** (2004) 105–128

[2] Garey, M.R., Johnson, D.S.: Computers and intractability. Freeman, San Francisco (1978)

[3] Kahng, A.B., Robins, G.: On optimal interconnections for VLSI. Kluwer Publsihers (1995)

[4] Caldwell, A., Kahng, A., Mantik, S., Markov, I., Zelikovsky, A.: On wirelength estimations for row-biased placement. In: Proceedings of international symposium on physical design, California. (1998) 4–11

[5] Korte, B., Prömel, H.J., Steger, A.: Steiner trees in vlsi layouts. Paths flows and VLSI layout, Springer (1990)

[6] Cheng, X., Huang, X., Li, D., Wu, W., Du, D.Z.: A polynomial-time approximation scheme for the minimum-connected dominating set in ad hoc wireless networks. Networks **42** (2003) 202–208

[7] Corneil, D.G., Olariu, S., Stewart, L.: Asteroidal triple-free graphs. SIAM J. Discrete Math. **10** (1997) 399–430

[8] Even, S., Pnueli, A., Lempel, A.: Permutation graphs and transitive graphs. J. ACM **19** (1972) 400–410

[9] Lekkerkerker, C.G., Boland, J.C.: Representation of a finite graphs by a set of intervals on the real line. Fund. Math. **51** (1962) 245–264

[10] Brandstädt, A., Lee, V.B., Spinrad, J.P.: Graph Classes: A Survey. SIAM Monographs on Discrete Mathematics and Applications, SIAM, Philadelphia (1999)

[11] Kloks, T., Müller, D.K.H.: On the structure of graphs with bounded asteroidal number. Graphs and Combinatorics **17** (2001) 295–306

[12] Balakrishnan, H., Rajaraman, A., Rangan, C.P.: Connected domination and steiner set on asteroidal triple-free graphs. In Dehne, F.K.H.A., Sack, J.R., Santoro, N., Whitesides, S., eds.: WADS. Volume 709 of Lecture Notes in Computer Science., Springer (1993) 131–141

[13] Ausiello, G., Crescenzi, P., Gambosi, G., Kann, V., Marchetti-Spaccamela, A., Protasi, M.: Complexity and Approximation. Springer, Heidelberg (2003)

[14] Motwani, R.: Lecture notes on approximation algorithms - volume i. Dept. of Comp. Sc., Stanford University, CA 94305-2140 (1992)

[15] Habib, M.: Substitution des structures combinatoires, theorie et algorithmes. these d'etat, Paris VI (1981)

[16] Kratsch, D., Spinrad, J.: Between o(nm) and o(n^α). In: SODA. (2003) 709–716