Manifold Learning and it's application

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SE367

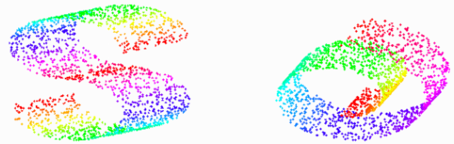
Outline

What is Manifold?

Definition

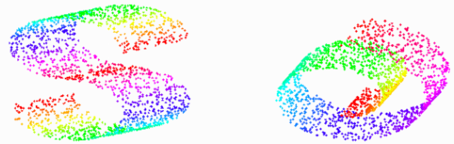
- Manifold is a mathematical space that on a small enough scale resembles the Euclidean space of a specific dimension.
- Hence a line and a circle are one-dimensional manifolds, a plane and sphere (the surface of a ball) are two-dimensional manifolds.

Introduction contd.



"S"-shape "Swiss roll"

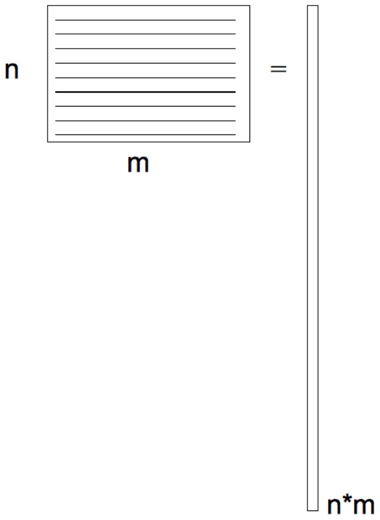
Introduction contd.



"S"-shape "Swiss roll"

Both are two-dim. data embedded in 3D

Images as Vectors



Why reduce Dimension?

- Curse of Dimensionality.
- Image data(each pixel), spectral coefficients, Text categorization(frequencies of phrases in a document), genes, many more.
- Practical data lie in subspace which is low dimensional.

Dimension reduction techniques

Linear methods

- Principal Component Analysis(PCA)
- Multidimensional Scaling(MDS)

Non-linear Methods

- Locally Linear Embedding (Roweis and Saul)
- IsoMap (Tenenbaum, de Silva, and Langford)

PCA-Principal Component Analysis

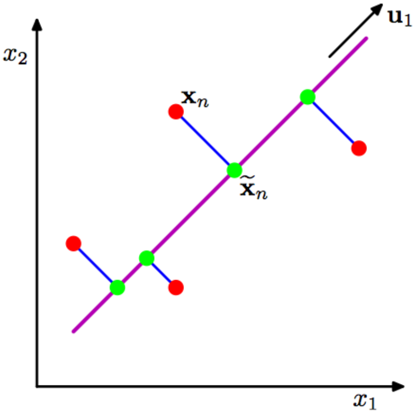
PCA

- Transforms possibly correlated variables into a smaller dimensional uncorrelated variables called principal components.
- Find linear subspace projection P which preserves the data locations (under quadratic error)

Algorithm

- Evaluate Covariance matrix $C = (x - \mu_x)(x - \mu_x)^T$
- Simple eigenvector(V_q) solution
- Top q eigenvectors of X$CC^T X^T$ V_q=[$v_1, \ldots v_q$] is a basis for the q-dim subspace Locations given by V_q^TXC

Example



PCA - Algorithm

n is number of data points, D is dimension of input, d is output dimension.

Data = $\{X_1, X_2, \ldots, X_n\}$, $X_i \epsilon R^D, i = 1, \ldots n$

goal is

$y_i \epsilon R^d$, i =1, . . . , n d << D

minimize reconstruction error

$min_{P \epsilon R^{mxm}} \sum_{i=1}^{n} ||X_i - PX_i||^2$ where P = U U^T $U \epsilon R^{Dxd}$

maximize the variance

$max \sum ||y_i - y_j||^2$

Then $y_i = U^T X_i$

Multidimensional Scaling

Given “pre-distances” D
Find Euclidean q-dim space which “preserves” these distances
Solution is given by the eigenstructure of Top q eigenvectors
This is exactly the same solution as PCA

Deficiencies of Linear Methods

- Data may not be best summarized by linear combination of features
- Example: PCA cannot discover 1D structure of a helix

How brain store these?

Every pixel?
Or perceptually meaningful structure?

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Every pixel?
Or perceptually meaningful structure? : Up-down pose Left-right pose Lighting direction
So, our brain successfully reduced the high-dimensional inputs to an intrinsically 3-dimensional manifold!

LLE and Isomap

"Local" relationships: Two solutions which preserve local structure:

LLE

Locally Linear Embedding (LLE) :

- Change to a local representation (at each point)
- Base the local rep. on position of neighboring points

Isomap

IsoMap:

- Estimate actual (geodesic) distances in p-dim. space
- Find q-dim representation preserving those distances

Both rely on the locally flat nature of the manifold

- How do we find a locality in which this is true?

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    - k-nearest-neighbors

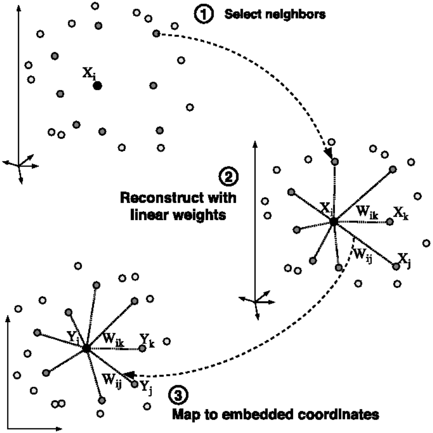
Both rely on the locally flat nature of the manifold

- How do we find a locality in which this is true?
    - k-nearest-neighbors
    - epsilon-ball

Local Linear Embedding

- Select a local neighborhood
- Change each point into a coordinate system based on its neighbors
- Find new (q-dim) coordinates which reproduce these local relationships

LLE by image



1 Select neighbors
X_i
2 Reconstruct with linear weights
X_i
W_{ik}
X_k
W_{ij}
X_j
3 Map to embedded coordinates
Y_i
W_{ik}
Y_k
W_{ij}
Y_j

Locally Linear Embedding

Find new (q-dim) coordinates which reproduce these local coordinates.
This can be solved using the eigenstructure as well: We want the min. variance

Locally Linear Embedding

minimize $\sum_{i=1}^{n} |X_i - \sum_{j=1}^{k} W_{ij} X_j|^2$ W is n x n sparse matrix

minimize $\sum_{i=1}^{n} |y_i - \sum_{j=1}^{k} W_{ij} y_j|^2$
or $\sum_{i=1}^{n} \sum_{j=1}^{n} M_{ij} y_i^T y_i$ M = n x n matrix

M = (I - W)$^T (I - W)$

Isomap

Distance Matrix

Build a graph with K-nearest neighbor or ϵ nearest neighbor.

Geodesic Distance

Infer other interpoint distances by finding shortest paths on the graph (Dijkstra's algorithm)

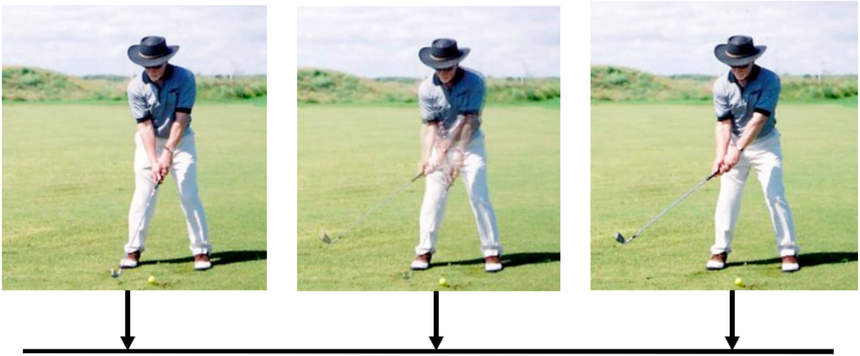
MDS

Apply MDS to embed the nearest neighbors.

an Application

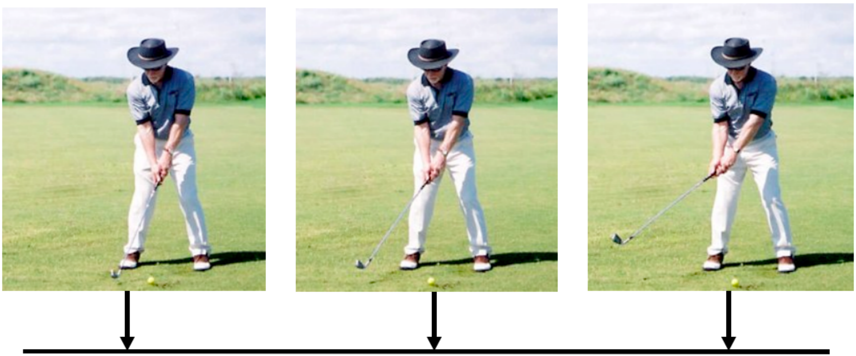
images from www.golfswingphotos.com

an Application



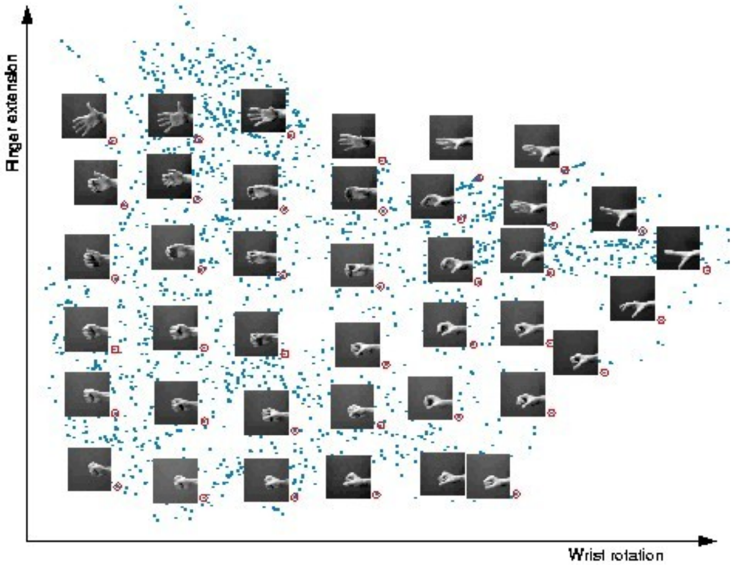
Linear Interpolation

an Application



Interpolation By Manifold

another Application(Isomap)



other Applications

Example

- Pose estimation
- image denoising
- missing data interpolation

Image References

- Learning-based Methods in Vision Alexei Efros, CMU, Spring 2007
- Nonlinear Dimensionality Reduction by Locally Linear Embedding Sam T. Roweis and Lawrence K. Saul

Further Reading

- **MDS :** Encyclopedia of Cognitive Science Multidimensional Scaling Mark Steyvers
- **PCA:** Jonathon Shlens, A Tutorial on Principal Component Analysis or refer Wikipedia
- **ISOMAP:** A Global Geometric Framework for Nonlinear Dimensionality Reduction Joshua B. Tenenbaum, Vin de Silva, John C. Langford
- **LLE:** Nonlinear Dimensionality Reduction by Locally Linear Embedding Sam T. Roweis and Lawrence K. Saul

Further Reading

Thank you!

http://home.iitk.ac.in/~ndubey/thesis