CS 687 2025-26 Sem II : Homework 2

Due: April 24

1. Let X be a discrete random variable taking only finitely many values. Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary function. Then show that $H(f(X)|X) = 0$.

   Conversely, show that if a random variable Y is such that $H(Y|X) = 0$, then there is a function $g : \mathbb{R} \to \mathbb{R}$ such that $Y = f(X)$.

2. A sequence $X \in \{0,1\}^\infty$ is *rational* if it is of the form wv^∞ - *i.e.* a prefix w followed by infinite repetitions of the string v. Note that w may be the empty string. Show that for every rational sequence, there is a lower semicomputable martingale $d : \{0,1\}^* \to [0,\infty)$ which succeeds on it.

3. Let $d_0, d_1, d_2, \ldots$ be a computably enumerable sequence of lower semicomputable martingales. Then show that there is a single lower semicomputable martingale d such that if any d_i succeeds on a sequence X, then d also succeeds on X. [Hint: Take a convex combination of martingales.]

4. A probability measure on $\{0,1\}^*$ is a function $\nu : \{0,1\}^* \to [0,1]$ such that $\nu(\lambda) = 1$ and for every string w, we have $\nu(w) = \nu(w0) + \nu(w1)$. An example is the "uniform distribution" $\mu : \{0,1\}^* \to [0,\infty)$.

   Suppose ν is an arbitrary probability distributions on strings in the above sense. Then show that the function $f : \{0,1\}^* \to [0,\infty)$ defined by $f(w) = \nu(w)/\mu(w)$ is a martingale. (This is well-defined since $\mu(w) \neq 0$ for all strings w.)