CS 687 2026: Homework 1

1 Part 1: Preliminaries

1. Show that L and L^c are acceptable if and only if there is a Turing machine M which, for every string x, halts and accepts x if $x \in L$ and halts and rejects x if $x \notin L$. [10 points]

2. Show that every infinite computably enumerable language contains an infinite decidable subset. [10 points]

3. Call a set $A \subset \mathbb{N}$ *sparse* if for every $k \in \mathbb{N}$, among the first 2^k natural numbers, the set A contains at most k numbers.

   Consider the set of all sparse subsets of $\mathbb{N}$. Is this set countable or uncountable? Prove your claim. [10 points]

2 Part 2: Kolmogorov complexity

1. (Data Processing Inequality) Show that if x is any finite string, and $f : \Sigma^* \to \Sigma^*$ is any total computable function, then $C(f(x)) \leq C(x) + O(1)$. This says that you can never *increase* information through computation, you can either preserve it or decrease it. [10]

2. Let x be an arbitrary finite binary string. Do all permutations of x have the same plain Kolmogorov complexity of x? If so, prove your claim. Otherwise, construct a counterexample, and prove that your string has some permutation which has a significantly different Kolmogorov complexity. [10]

3. (Characteristic sequence of the halting problem) Let χ_H be the infinite binary sequence defined as follows. For any number i, the i^{th} bit of χ_H is 1 if the i^{th} Turing machine halts, and 0 otherwise. Since the Halting problem is uncomputable, χ_H is uncomputable. However, show that prefixes of χ_H are compressible: for all sufficiently large

n, show that $C(\chi_H[0 \ldots n-1]) \leq \log n + O(1)$, where $\chi[0 \ldots n-1]$ denotes the n-length prefix of χ_H. (This shows that even when a string is uncomputable, it may be highly compressible.) [10]

4. (Chaitin's Omega, modified version) Let P be a prefix-free set of strings, which form the domain of the universal prefix-free machine M. Let ω be an infinite binary sequence defined by

$$\omega = \sum_{\substack{p \in \mathbb{P}, \\ M(p)\downarrow}} \frac{1}{2^{|p|}}.$$

Show that prefixes of ω are incompressible: for all sufficiently large n, $C(\omega[0 \ldots n-1]) \geq n - O(1)$. (The purpose of this question is to show that there are uncomputable strings which are also incompressible, in contrast to the previous question.) [10]

5. A course meets for n lectures a semester. The course will have a "surprise" quiz (i.e. unannounced, and at some unpredictable point) during some lecture. A famous argument says that no such quiz can be surprising. (see The Unexpected Hanging Paradox on Wikipedia). This puzzle was popularized by Martin Gardner in the 1960s. Propose a resolution, using Kolmogorov complexity, that we can indeed have a "surprising" quiz. Define your notion of surprise, and then show that some days can have surprise quizzes according to this definition. [10]