Incompressibility $\to$ ML Test

Fix a prefix-free machine M. Consider

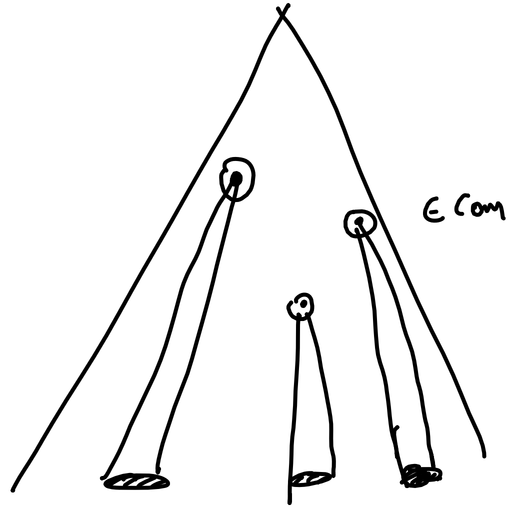
$$Comp_b = \{ x \in \Sigma^* \mid K(x) \leq |x| - b \}.$$

and $$R_b = \bigcup_{x \in Comp_b} [\![x]\!]$$ i.e. set of all infinite sequences with some b-compressible prefix.

(1) R_b is c.e. given b: We can dovetail over all programs and accept x as a member of $Comp_b$ when some program of length $\leq |x| - b$ halts & outputs x. (Otherwise, we run forever, but that is fine since we need only c.e.)

(2) $R_1, R_2, \ldots$ are uniformly c.e. in the index — easy to see since the above conversion from b to the program accepting Comp_b can be done by a program.

(3) <u>Claim.</u> $\mu(R_b) \leq 1/2^b$.



<u>Proof</u>

Let V_b be the largest prefix-free subset of Comp_b.

Then

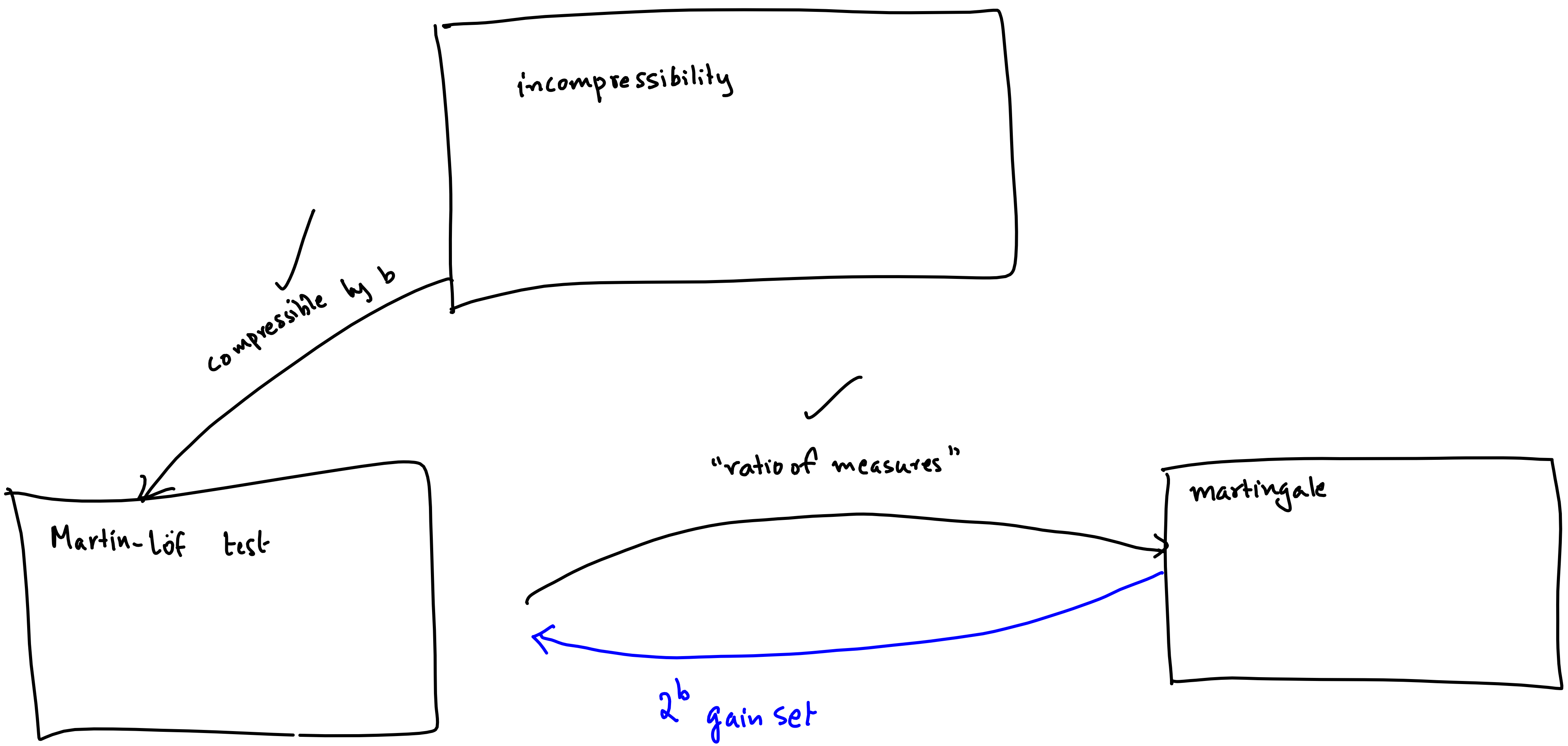
$$\sum_{x \in V_b} \frac{1}{2^{|x|}} = \mu(R_b).$$

Let $x \in V_b$. Pick the shortest program p_x for x. Then $|p_x| \leq |x| - b$, since $x \in V_b \subseteq \mathrm{Comp}_b$. Then $2^{-|p_x|} \geq 2^{+b} \cdot 2^{-|x|}$. Hence,

$$1 \overset{\text{Kraft inequality}}{\geq} \sum_{x \in V_b} \frac{1}{2^{|P_x|}} \overset{|P_x| \leq |x| - b.}{\geq} 2^b \left(\sum_{x \in V_b} \frac{1}{2^{|x|}} \right) = 2^b \cdot \mu(R_b)$$

Thus $\mu(R_b) \leq 2^{-b}$.

Hence, by (1), (2), (3), it follows that $R_1, R_2, \ldots$ is a Martin-Löf test.


incompressibility
compressible by b
Martin-Löf test
"ratio of measures"
martingale
2^b gain set