Def.

A sequence of open sets $G_1, G_2, \dots$ is called a Martin-Löf test if the sequence is uniformly c.e. and $\forall m \quad \mu(G_m) \leq 2^{-m}$. (denoted $\langle G_m \rangle_{m \in \mathbb{N}}$)

$X \in \{0,1\}^\infty$ fails the test if $X \in \bigcap_m G_m$ [observe : $\mu(\bigcap_m G_m) \leq \lim_{n \to \infty} 2^{-m} = 0$.]

Intuitively, G_m is identifying a statistical "defect" in X with degree m.

X is nonrandom if $\exists$ ML Test $G_1, G_2, \dots$ such that X fails the test.

Theorem [Martin-Löf, 1966]

There is a universal Martin-Löf test

Proof

Let
$$\begin{matrix} G^1_1, G^1_2, \ldots \\ G^2_1, G^2_2, \ldots \\ \vdots \end{matrix} = \langle\langle G^e_m \rangle_{m\in\mathbb{N}} \rangle_{e\in\mathbb{N}}$$

be a uniformly computably enumerable sequence of open sets such that $\forall e$ $\langle G^e_m \rangle_{m\in\mathbb{N}}$ is a Martin-Löf test, where every ML test appears as some row.

Now define, $\forall b \in \mathbb{N}$, $\mathcal{U}_b = \bigcup_{e\in\mathbb{N}} G^e_{e+b+1}$.

For example, for $b=2$,

$$U_2 = G^1_{1+2+1} \cup G^2_{2+2+1} \cup \cdots$$

$$= G^1_4 \cup G^2_5 \cup \cdots$$

$$\boxed{\begin{array}{ccccccc}
G^1_1 & G^1_2 & G^1_3 & G^1_4 & \cdots & & \\
G^2_1 & G^2_2 & G^2_3 & G^2_4 & G^2_5 & \cdots & \\
G^3_1 & G^3_2 & G^3_3 & G^3_4 & G^3_5 & G^3_6 & \cdots \\
\vdots & & & & & & U_2
\end{array}}$$

Now, U_b is a c.e. union of open sets. So U_b is c.e. open.
U_bs are uniformly c.e. Further, $\mu(U_b) \leq \sum_{e\in\mathbb{N}} \mu\left(G^e_{e+b+1}\right) \leq \sum_{e\in\mathbb{N}} 2^{-(e+b+1)} = 2^{-b}$

Suppose $Z \in \{0,1\}^\infty$ is not Martin-Löf random. Then $\exists e$ such that $Z \in \bigcap_m G^e_m$.
Thus, there is a row such that Z is inside every open set in that row.
Hence $Z \in U_b$ for all b. Thus $Z \in \bigcap_b U_b$.

Thus $\langle U_b \rangle_{b \in \mathbb{N}}$ is a <u>universal</u> ML test – it contains all non randoms. $\square$.

$$\boxed{\begin{aligned} \mu\left(\bigcap_{b=1}^{\infty} U_b\right) &\leq \mu\left(\bigcap_{b=1}^{n} U_b\right) \\ &\leq \mu(U_b) \\ &\leq 2^{-b} \end{aligned}}$$

<u>Def</u>
$X \in \{0,1\}^\infty$ is called **Martin-Löf random** if $X \notin \bigcap_b U_b$.

Since $\bigcap_b U_b$ is the "largest" constructive measure 0 set, we can say that the set of Martin löf randoms is the "smallest" co-constructive measure 1 set.

Example:

If $Z \in \{0,1\}^\infty$ is computable, then Z is not ML random.

Proof.

Let M be a machine such that $\forall i \in \mathbb{N}$ $M(i) = Z[i]$. Such a machine exists since Z is computable.

Define $\forall m$ $G_m = \{ X \in \{0,1\}^\infty \mid X[0 \ldots m-1] = Z[0 \ldots m-1] \}$, the set of all infinite sequences with the same m-length prefix as Z.

denoted usually by $[[Z[0 \ldots m-1]]]$

Then, $\mu(G_m) = 2^{-m}$. Further, $Z \in G_m$. The sequence $\langle G_m \rangle_{m \in \mathbb{N}}$ is a uniformly c.e. sequence of open sets. Thus $Z \in \bigcap_m G_m$, proving that Z is not an ML random.

Equivalence b/w Martingale definition of randomness & ML Test definition of randomness.

<u>Recall</u>: X is not ML random if there is a lower semicomputable martingale $d : \Sigma^* \to [0,\infty)$ such that $\forall N \; \exists n \; d(X[0 \ldots n-1]) > N$.

We now show that

① if there is a Martin Löf test that captures a sequence X, then there is a lower semicomputable martingale that succeeds on X.

② Converse of ①.

ML Test $\Rightarrow$ martingale.

Let $\langle U_b \rangle_{b\in\mathbb{N}}$ be the universal ML test. Any sequence X is non-ML random iff $X \in \bigcap_b U_b$.

For each $b\in\mathbb{N}$, define $d_b : \Sigma^* \to [0,\infty)$ by

$$d_b(x) = \mu\left(U_b \cap [\![x]\!] \right) \cdot 2^{|x|}$$

(For the $[\![\,]\!]$ notation, see page 5.) (Intuition: d_i is the ratio of a normalized uniform distribution to the actual uniform distribution.)

(1) $d_b(\lambda) = \mu(U_b \cap [\![\lambda]\!]) \cdot 2^{|\lambda|} = \mu(U_b) \leq 1.$

(2)

$$\frac{d_b(x0) + d_b(x1)}{2} = \frac{\left[\mu\left(U_b \cap [[x0]]\right) * 2^{|x0|} + \mu\left(U_b \cap [[x1]] * 2^{|x1|}\right)\right]}{2}$$

$$= \frac{\left[\mu\left(U_b \cap [[x0]]\right) + \mu\left(U_b \cap [[x1]]\right)\right]}{2} * 2^{|x|+1}$$

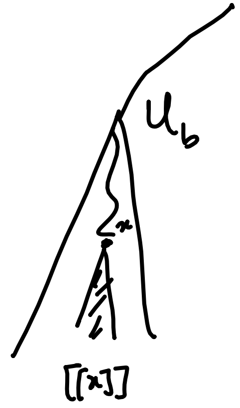
$$= \left[\mu\left(U_b \cap [[x0]]\right) + \mu\left(U_b \cap [[x1]]\right)\right] * 2^{|x|}$$

$$= \mu\left(U_b \cap [[x]]\right) * 2^{|x|}$$ since the disjoint union of $[[x0]]$ and $[[x1]]$ is precisely $[[x]]$

$$= d_b(x).$$

Thus d_b is a martingale.

For every $X \in \{0,1\}^{\omega}$, for every prefix x of X inside $\mathcal{U}_b$.



$$d_b(x) = \mu\left(\mathcal{U}_b \cap [[x]]\right) 2^{|x|}$$

$$= \mu([[x]]) \cdot 2^{|x|}$$

$$= 1.$$

Then $d = \sum_{b \in \mathbb{N}} d_b$ is a lower semicomputable martingale that succeeds on every $X \in \bigcap_b \mathcal{U}_b$

Incompressibility of prefixes

$t(x) = |x| - K(x \mid |x|)$

universal

=

=

Martinlof tests

universal

=

Martingales

universal

Unrelated

$$\bigcap_{b=1}^{\infty} \left[0, \tfrac{1}{2^b}\right] = \left[0, \tfrac{1}{2}\right] \cap \left[0, \tfrac{1}{4}\right] \cap \cdots$$

$$= \{0\}.$$