$d : \Sigma^* \to [0,\infty)$ is called lower semicomputable if there is a function

$\hat{d} : \Sigma^* \times \underbrace{\mathbb{N}}_{\text{patience parameter}} \to \mathbb{Q} \cap [0,\infty)$ such that

(1) [Monotonically Increasing] $\forall w \in \{0,1\}^* \ \forall n \in \mathbb{N}$ $\hat{d}(w,n) \le \hat{d}(w,n+1) \le d(w)$

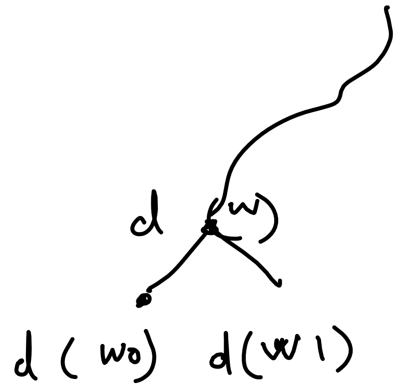
(2) [Convergence] $\forall w \in \{0,1\}^*$ $\lim_{n\to\infty} \hat{d}(w,n) = d(w)$.

We will consider lower semicomputable martingales.

Martingale:

(1) $d(\lambda) < \infty$

(2) $\forall w \in \{0,1\}^*$ $\quad d(w) = \dfrac{d(w0) + d(w1)}{2}$



d succeeds on $X \in \{0,1\}^{\omega}$ if $\forall N \ \exists n \in \mathbb{N}$ $\quad d(X[0\ldots n-1]) > N$.

$x \longrightarrow x,$

(1) 0^{∞}: Martingale that succeeds on this:

Define $d: \{0,1\}^* \to [0,\infty)$ by

(1) $d(\lambda) = 1$.

(2) For $w \in \{0,1\}^*$ $\quad d(w0) = 2\,d(w)$

$d(w1) = 0$ $\quad \Rightarrow \quad \dfrac{d(w0)+d(w1)}{2} = d(w)$.

Now $d(0^n) = 2^n$ by induction.

ToShow: d is computable, hence lower semicomputable.

$\lim_{n\to\infty} d(0^n) = \infty.$

So d succeeds on 0^∞.

(Intuition: 0^∞ is not random)

2) Suppose X is an infinite binary sequence with

$$\lim_{n\to\infty} \frac{X[0] + \dots + X[n-1]}{n} = \lim_{n\to\infty} \frac{\text{Number of 1s in } X[0],\dots,X[n-1]}{n}$$

$= 0.75.$

Define $d: \{0,1\}^* \to [0,\infty)$ by

(1) $d(\lambda) = 1.$

(2) For any $w \in \{0,1\}^*$

$$\left.\begin{aligned} d(w0) &= 2 \times \tfrac{1}{4} \times d(w) \\ d(w1) &= 2 \times \tfrac{3}{4} \times d(w) \end{aligned}\right\} \quad \frac{d(w0)+d(w1)}{2} = \left(\tfrac{1}{4}+\tfrac{3}{4}\right) d(w) = d(w)$$

d is computable, hence lower semicomputable.

Now, we show that d succeeds on any such X.

Approx:

$$d(X[0..n-1]) \approx \left(2\times\frac{3}{4}\right)^{\frac{3}{4}n} \times \left(2\times\frac{1}{4}\right)^{\frac{1}{4}n}$$

$$= \Big[\underbrace{(1.5)^3 \times (0.5)}\Big]^{\frac{n}{4}}$$

$$\geq \left[(2.25\times 0.5)\times 1.5\right]^{n/4}$$

$$> [1.5]^{n/4} \longrightarrow \infty \text{ as } n\to\infty.$$

(3) Suppose X is a sequence that does not have 0010

Define $d : \{0,1\}^* \to (0,\infty)$ by

1) $d(\lambda) = 1$.

2) For any string w, define

$$d(w0000) = 16 \times \frac{1}{15} \times d(w)$$

$$d(w0001) = 16 \times \tfrac{1}{15} \times d(w).$$

$$\boxed{d(w0010) = 0}$$

$$d(w0011) = 16 \times \tfrac{1}{15} \times d(w)$$

$$\vdots$$

$$d(w1111) = 16 \times \tfrac{1}{15} \times d(w)$$

Kolmogorov Inequality

$$\Pr\left[x \in \Sigma^n \mid d(x) > N\right] < d(\lambda)/N$$

can be generalized to

Let $S \subseteq \Sigma^*$ be a prefix-free set. Then $P\left[x \in S \mid d(x) > N\right] < \frac{d(\lambda)}{N}$.

Informal

Now, we can show that

$$\Pr\left[X \in \{0,1\}^\infty \mid \forall N \exists n \quad d(X[0 \ldots n-1]) > N\right] \lesssim \lim_{N \to \infty} \frac{d(\lambda)}{N} = 0.$$

Every martingale succeeds only a probability 0 set.