Theorem

$\forall x \in \Sigma^*$ $\quad K(x) \leq -\log m(x) + c$, where $m(x) = \sum_{\substack{p \in \mathcal{P} \\ p(\lambda) = x}} \frac{1}{2^{|p|}}$.

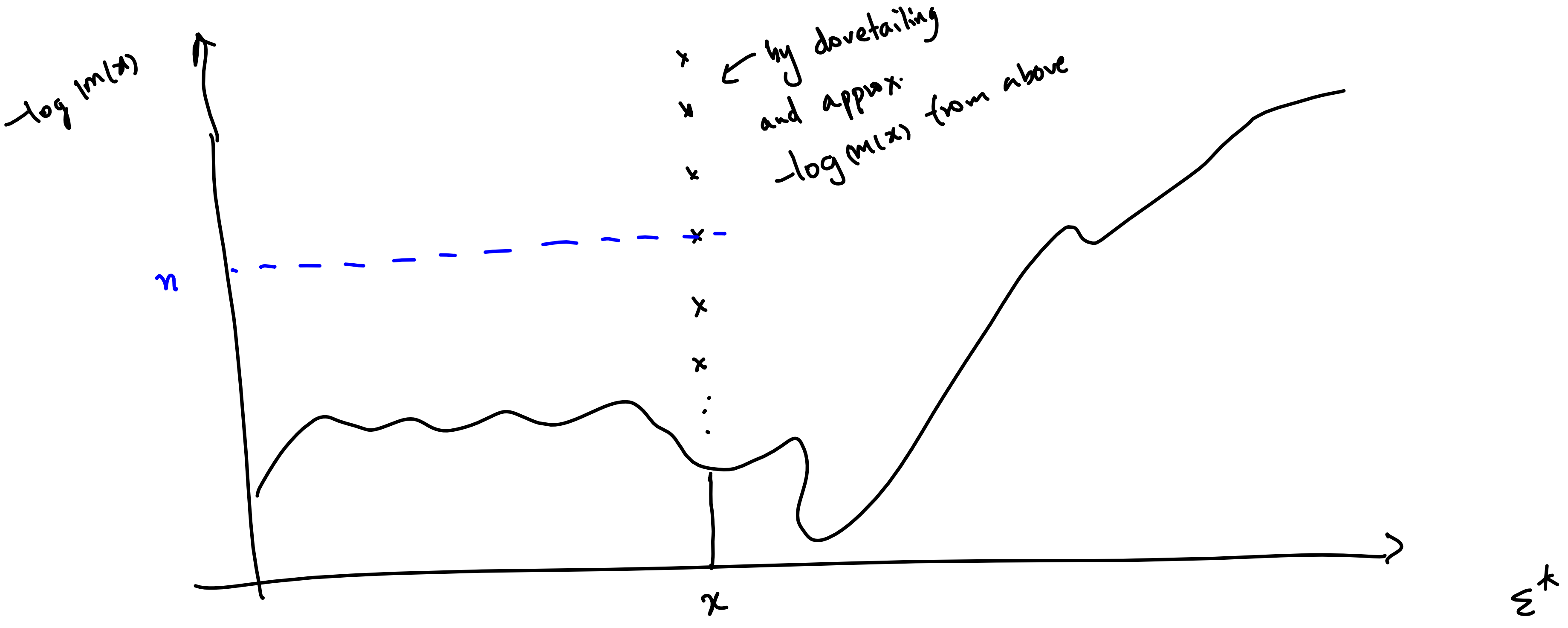
Proof

We form a request set.

$$\{ (x, s_n) \mid -\log m(x) < n+1, \; x \in \Sigma^*, \; n \in \mathbb{N} \}$$

Note $\{ (x, s_{-\log m(x)}) \mid x \in \Sigma^* \}$: this satisfies Kraft inequality, but is not a c.e. set.

But since $m(x)$ can be computably approximated from below, $-\log m(x)$ can be computably approximated from above.



By dovetailing, it is possible to show that

$$L = \{ (x, s_n) \mid x \in \Sigma^*, n \in \mathbb{N}, -\log m(x) < n+1 \}$$ is computably enumerable

Kraft inequality:

$$\sum_{(x, s_n) \in L} \frac{1}{2^n} = \sum_{x \in \Sigma^*} \left(\sum_{-\log m(x) - 1 < n} \frac{1}{2^n} \right) = \sum_{x \in \Sigma^*} \left(\frac{1}{2^{-\log m(x) - 1}} + \frac{1}{2^{-\log m(x) - 2}} + \cdots \right)$$

$$\le \sum_{x \in \Sigma^*} m(x) \le 1.$$

Thus by Levin's algorithm, all requests will be satisfied. For each x, the shortest request satisfies:

$$|n| \leq -\log \mathrm{m}(x) + 1$$

Hence Levin's prefix code witnesses

$$K(x) \leq -\log \mathrm{m}(x) + O(1). \qquad \square$$

<u>Aside:</u> If X is a random variable, then $-\log p(X)$ is called the "surprisal" of X.

Symmetry of Information.

$$K(x,y) \stackrel{?}{=} K(x) + K(y|x) + O(1) \stackrel{?}{=} K(y) + K(x|y) + O(1)$$

Suppose this holds. Then

$$\underbrace{K(x) - K(x|y)} = K(y) - K(y|x) + O(1)$$

This term is called mutual information and is written $I(x;y)$.

Symmetry of Information $I(x;y) = I(y;x)$.

$$s(a,b) = a.$$

$$q_p = p.s$$

Lemma There is a constant c_1 such that

$$\forall x \in \Sigma^* \quad 1m(x) \geq \frac{1}{2^{c_1}} \sum_{\substack{p \in P,\ y \in \Sigma^* \\ U(p) = (x,y)}} \frac{1}{2^{|p|}}$$

Proof.

Let q_p be a program that runs p, and if it produces a pair of strings, outputs its first coordinate.

Then $U(q_p) = x$ if $p(\lambda) = (x, \cdot)$

$$|q_p| \leq |p| + c_1.$$

Now

$$1m(x) = \sum_{\substack{q \in P \\ q(\lambda) = x}} \frac{1}{2^{|q|}} \geq \sum_{\substack{p \in P,\ y \in \Sigma^* \\ U(p) = (x,y)}} \frac{1}{2^{|q_p|}} = \frac{1}{2^{c_1}} \sum_{\substack{p \in P,\ y \in \Sigma^* \\ U(p) = (x,y)}} \frac{1}{2^{|p|}}.$$

Theorem [Symmetry of Information]

$\exists c \quad \forall x, y \in \Sigma^*$

$$K(x,y) \geq K(x) + K(y \mid x, K(x)) + O(1)$$

Proof

Strategy: Form a request set.

The inequality is:

$$K(y \mid x, K(x)) \leq K(x,y) - K(x) + O(1).$$

Let R be the Turing Machine which when given input $z = (x, S_n)$, does the following:

If there is a program of length n which produces x, then

$R(z)$ enumerates:

$$L(R(z)) = \{ (y, \quad |p| - n + c) \mid U(p) = (x, y) \}.$$

When $n = K(x)$, then $L(R(z))$ is a request set. (Proof in notes)

$$L(R(z)) = \{ (y, \ |p| - K(x) + c \mid U(p) = (x, y) \}$$

When $p = K(x, y)$, we have the request is $(y, K(x,y) - K(x) + c)$.