

Levin's Coding Theorem

$$\forall x \in \Sigma^* \quad K(x|z) = -\log m(z) + O(1).$$

Proof

Greedy algorithm allocating the leftmost available node in the infinite binary tree

We now argue that if $R(z)$ is a c.e. request set, then the algorithm never gets stuck.

Formal argument: please read the notes.

Idea: Illustration

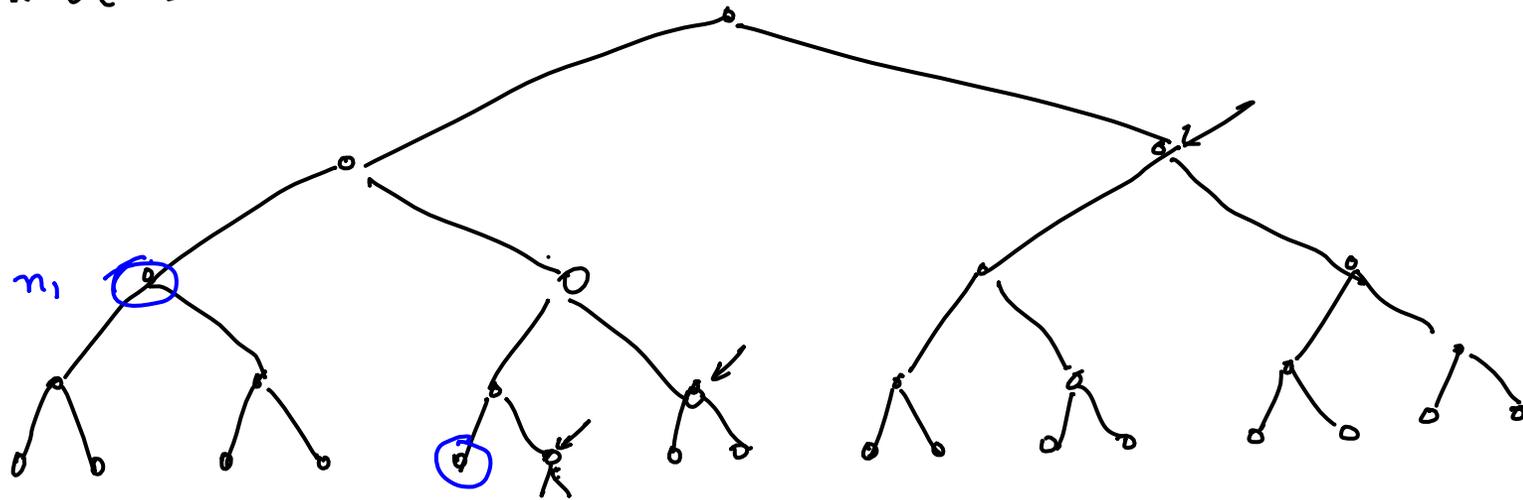
At every stage i , we have allocated $(i-1)$ strings into the prefix code.

Let $[w]$ denote the set of all binary sequences starting with w .

$$\begin{aligned} (C^{(1)})^c &= [0100] \cup [0101] \cup \dots \cup [1111] \\ &= [01] \cup [1] \end{aligned}$$

$$(C^{(2)})^c = [0101] \cup [011] \cup [1] \quad n_1$$

$$\begin{aligned} n_1 &= 2 & C^{(1)} &= [01] \\ n_2 &= 4 & C^{(2)} &= [00] \cup [0100] \end{aligned}$$



At any i , if $C^{(i)}$ denotes the allocated part of the infinite binary tree,

then $(C^{(i)})^c$ can be written as a finite union of $[-]$ terms,

where the string lengths are strictly decreasing.

Now assume that for some length n , no more strings of length n are available. But this means that every string w_i in the minimal

term for $C^c = \bigcup_{i=1}^k [w_i]$, where $|w_1| > |w_2| > \dots > |w_k|$, satisfies $|w_i| \geq n+1$.

(equivalently, $|w_k| \geq n+1$).

$$\frac{1}{2^{|w_1|}} + \frac{1}{2^{|w_2|}} + \dots + \frac{1}{2^{|w_k|}} \leq \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} + \dots$$
$$< \sum_{i=1}^{\infty} \frac{1}{2^{n+1+i}} = \frac{1}{2^n}.$$

Hence the ^(requester) adversary's budget $\sum_{j=1}^{\infty} \frac{1}{2^{n_i+j}} < \frac{1}{d^n}$ otherwise it will violate the Kraft inequality. Hence the requester cannot issue any more request with code length $\leq n$.

Hence the algorithm never gets stuck. \square .

By handcoding z , we get

Corollary

If L is a c.e. request set, $\exists c_L$ s.t. $\forall (x, s_n) \in L,$

$$K(x) \leq n + c_L.$$

$\{(x, \lceil -\log m(x) \rceil) \mid x \in \Sigma^*\}$ does not quite work as a request set.

Clearly

$$\begin{aligned} \sum_{x \in \Sigma^*} \frac{1}{2^{\lceil -\log m(x) \rceil}} &< \sum_{x \in \Sigma^*} \frac{1}{2^{-\log m(x)}} \\ &= \sum_{x \in \Sigma^*} m(x) \\ &\leq 1. \end{aligned}$$

So it could form a request set, since the codelengths satisfy the Kraft inequality. But it is not c.e.

$m(x)$ is comp. approximable from below

$\Rightarrow \frac{1}{m(x)}$ " above

" above

$\Rightarrow \log\left(\frac{1}{m(x)}\right)$

" above

$\Rightarrow -\log m(x)$

$(-\log$

