

# Levin's Coding Theorem

$$\forall x \in \Sigma^* \quad K(x) \leq -\log m(x) + O(1)$$

$$\frac{\text{Recall}}{m(x)} = \sum_{\substack{p \in \mathcal{U} \\ p(\lambda) = x}} \frac{1}{2^{|p|}}$$

Idea:

Given a list of strings  
assign a prefix code

& the "probability" of each string,  
which satisfies the theorem.

Change this slightly:

Given  $m(x)$ , consider  $-\log m(x)$ .

If we assign some code to  $x$  having length  $-\log m(x)$ , then,  
its contribution to the Kraft sum =  $\frac{1}{2^{-\log m(x)}} = m(x)$ .

So: we are given a sequence of strings & requests <sup>similar to</sup>  $(-\log m(x))$  for  
prefix code lengths for each string.

Goal: Construct a prefix code that satisfies all the demands.

Restrictions: the requests are algorithmically generated.

Def. An indexed enumerator is a Turing machine  $M$  that takes  
a string  $z$  and enumerates a c.e. language.

$M(x, 0)$        $M(x, 1)$        $M(x, 2)$       ...       $\lambda^{\text{th}}$  c.e. language.

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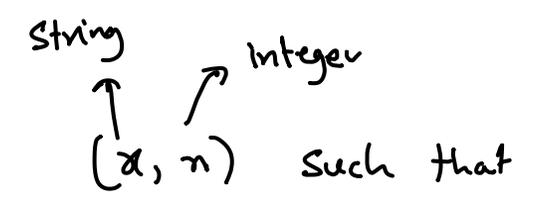
$M(0, 0)$        $M(0, 1)$        $M(0, 2)$       ...       $0^{\text{th}}$  c.e. language

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⋮

Definition A request set  $L$  is a collection of pairs  $(x, n)$  such that

$$\sum_{(x, n) \in L} \frac{1}{2^n} \leq 1.$$



Definition A conditional requester is an indexed enumerator  $R$  such that  $\forall z \in \Sigma^*$   $R(z)$  enumerates a request set.

e.g.

$$R(x) = \{ (x, 1), (0, 2), (1, 3), (00, 4), \dots \}$$

$$R(0) = \{ (0, 1), (x, 2), (1, 4), (00, 3), \dots \}$$

□

Theorem For every conditional requester  $R$ ,  $\exists$  constant  $c_R$  such that  $\forall z \in \Sigma^*$

and for every  $(x, n) \in L(R(z))$ , we have

$$|x| \leq n + c_R$$

(i.e. every request can be satisfied)

"greedily assign the leftmost free node of length  $n_i$ ."

## Proof

Sketch of the algorithm:

1. Input  $(i, \overline{z})$  <sup>integer</sup> string used for  $R(z)$ .

2.  $j=0$

3. for  $j=0$  to  $i$  do

3.1) Let  $R(z)$ 's  $j$ th request be  $(x_j, n_j)$

3.2) Find the leftmost free node in the binary tree of length  $j$ . Let that be  $c_{n_j}$ . Then assign  $c_{n_j}$  as the code for  $x_j$ .

3.3)  $j \leftarrow j+1$ .

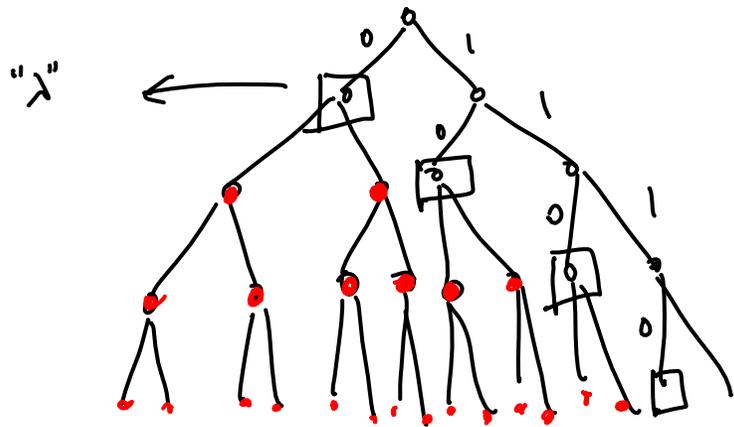
4. Output  $C_{n_i}$ .

$$R(z) = \{(x_0, n_0), (x_1, n_1), \dots\}$$

Suppose  $R(z)$  enumerates the ce. request set

$$\left\{ \left( \underset{\substack{\uparrow \\ j=0}}{\lambda}, 1 \right), \left( \underset{\substack{\uparrow \\ j=1}}{0}, 2 \right), \left( \underset{\substack{\uparrow \\ j=2}}{1}, 3 \right), \left( \underset{\substack{\uparrow \\ j=i=3}}{00}, 4 \right), \dots \right\}$$

Say  $i = 3$



$\lambda$	:	code	$\emptyset$
$0$	:	code	$10$
$1$	:	code	$110$
$00$	:	code	$1110$