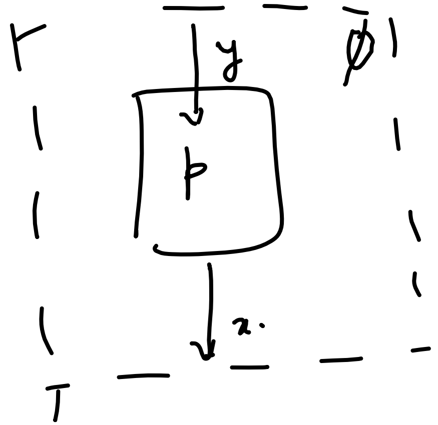
Recall

$$C_\phi(x|y) = \min\left(\{\, |p| \; : \; \phi(\langle y,p\rangle) = x \,\} \cup \{\infty\}\right)$$



$$C_\phi(x) \triangleq C_\phi(x|\lambda)$$

Fact:

$$C_\varphi(x|y) \le C_\varphi(x) + O(1)$$

Any Universal Partial Computable function essentially computes the same value of complexity for every x, as the following shows.

Theorem [Invariance Theorem]

There is a universal partial computable function $\psi : \Sigma^* \dashrightarrow \Sigma^*$ such that for any partial comp. function $\varphi : \Sigma^* \dashrightarrow \Sigma^*$, there is a constant c_φ such that for any x & y,

$$C_\psi(x|y) \leq C_\varphi(x|y) + c_\varphi .$$

Proof

Let ψ be the partial comp. function computed by the U.T.M. $\mathcal{U}$.
Such that $\mathcal{U}(\langle s_n, y, p\rangle)$ computes the n^{th} partial computable function φ_n on input $\langle y, p\rangle$. Then we have

$$\psi(\langle s_n, y, p\rangle) = \varphi_n(\langle y, p\rangle).$$

The only additional input to ψ is s_n, which takes at most $2|s_n| + 2$ bits.
Taking smallest p, we have

$$C_\psi(x|y) \leq C_{\varphi_n}(x|y) + \underbrace{2|s_n| + 2}.$$

$\square$

Def:

For this fixed universal partial function ψ,

$$C(x|y) \triangleq C_{\psi}(x|y)$$

$$C(x) \triangleq C_{\psi}(x|\lambda).$$

They will be called the plain Kolmogorov Complexity of x given y, and the plain KC of x, respectively.

Lemma

$\forall x \qquad C(x) \leq |x| + c.$

Proof

Consider

0. Input y
1. print x.

Since this program outputs x given λ, the shortest program for x given λ, must be shorter.

$$C(x) \leq |x| + c.$$