Theorem

For any set A, $\nexists\, f : A \xrightarrow[\text{onto}]{} \mathcal{P}(A)$.

Proof

Let $f : A \xrightarrow[\text{onto}]{} \mathcal{P}(A)$, where A is arbitrary.

Then consider the set

$$S = \{ i \in A \mid i \notin f(i) \}$$

Then note that for any $j \in \mathbb{N}$, $j \in S \Rightarrow j \notin f(j)$ and $j \notin S \Rightarrow j \in f(j)$. Hence, S cannot be $f(j)$.
Since this holds for all $j \in S$, we conclude that $S \notin \text{range}(f)$. $\square$

e.g. when $A = \mathbb{N}$

Let $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$.

For any subset $B \subseteq \mathbb{N}$, represent it as a "bitmap"

i^{th} bit is 1 if $i \in B$
0 o/w.

e.g. $B = \{2, 3, 4\}$

0	1	2	3	4	5	6	...
0	0	1	1	1	0	0	...

Suppose $f : \mathbb{N} \xrightarrow[\text{onto}]{} \mathcal{P}(\mathbb{N})$

$f(0)$	$\boxed{0}$	1	1	1	0	0	0	...
$f(1)$	0	$\boxed{1}$	0	1	0	....		
$f(2)$	1	0	$\boxed{0}$	1	0	...		

Consider the set $\{0, 2, \ldots$

i.e. i is in the above set iff i is not in $f(i)$ - i.e. i^{th} bit is 0 in $f(i)$.

$|\mathcal{P}(\mathbb{R})|$

$|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$

$|\mathbb{N}|$

What about $P \neq NP$?

Unfortunately, Baker, Gill, Solovay '75: $\exists A \subseteq \mathbb{N}$ such that $P^A \neq NP^A$

$\exists B \subseteq \mathbb{N}$ such that $P^B = NP^B$.

Turing Computability

A Turing Machine M is a 7 tuple $(Q, \Sigma, \Gamma, \delta, q_0, q_a, q_r)$

where

Q : finite set of states

Σ : finite set of letters. (input alphabet)

$\Gamma \supsetneq \Sigma$: finite set of letters, containing $B \notin \Sigma$, (tape alphabet)

$\delta : \Gamma \times Q \setminus \{q_a, q_r\} \longrightarrow \Gamma^* \times Q \times \{L, R\}$

$q_0 \in Q$: initial state

$q_a \in Q$: accept state

$q_r \in Q$: reject state.

A language L is a set of strings. $L \subseteq \{0,1\}^*$

L is (Turing) acceptable if $\exists$ Turing Machine M such that $\forall x \in L$ $M(x)$ enters q_a & halts, $\forall x \notin L$, $M(x)$ does not enter q_a.

L is decidable if L, and $\{0,1\}^* \setminus L$ are acceptable.