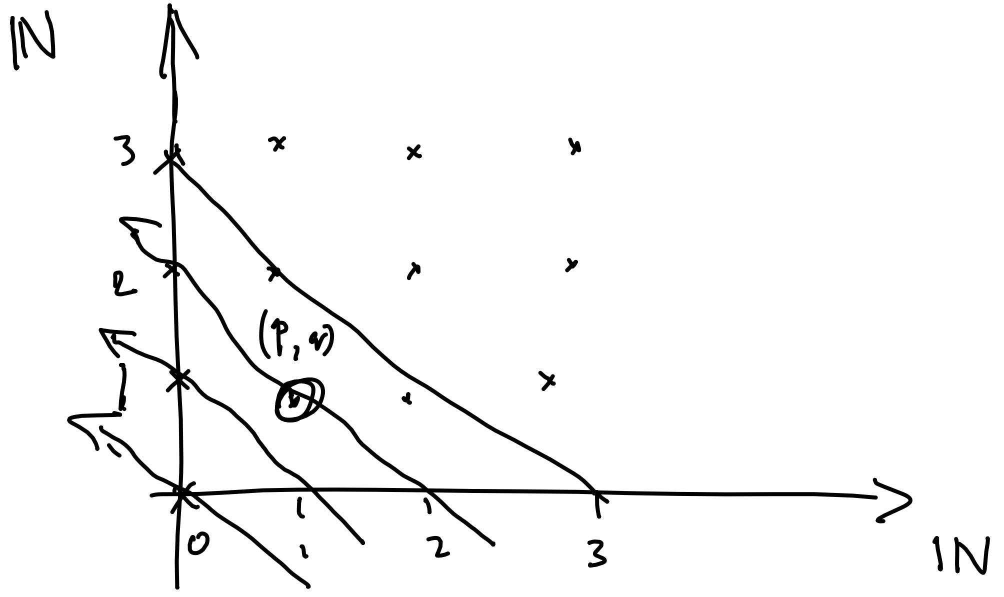
Theorem

$\mathbb{N} \times \mathbb{N}$ is countable.

$\mathbb{N} \times \mathbb{N} = \{(p,q) \mid p, q \in \mathbb{N}\}$

(**Lemma.** A countably infinite union of finite sets is countably infinite.).

Proof of Theorem



We need a function

$$f : (\mathbb{N} \times \mathbb{N}) \xrightarrow{1-1} \mathbb{N}$$

Define $f(p,q)$ by

$$\underbrace{\left[\sum_{j=0}^{(p+q-1)} (j+1)\right]}_{\substack{\text{number of} \\ \text{points on previous} \\ \text{diagonals}}} + \underbrace{(q+1)}_{\substack{\text{index of } (p,q) \\ \text{on the diagonal} \\ \text{containing } (p,q)}} = \left[\sum_{j=1}^{p+q} j\right] + (q+1).$$

It can be shown that f is 1 to 1. (see notes). $\blacksquare$

The technique is known as <u>dovetailing</u>.

Corollary

For any fixed positive integer k, $\mathbb{N}^k$ is countable.

Corollary

$$\mathbb{N}^* = \bigcup_{k=0}^{\infty} \mathbb{N}^k \quad \text{is countable}$$

[The set of all finite length tuples of natural numbers.]

$\mathbb{N}^\infty =$ (later)

Corollary $\mathbb{Q}$ is countable [Cantor]

Proof Let p/q be a rational in lowest form (i.e., $\gcd(p,q)=1$). Then define $f(p/q) = (p,q)$. This a 1-1 mapping from $\mathbb{Q}$ to $\mathbb{N}\times\mathbb{N}$. Since $\mathbb{N}\times\mathbb{N}$ is countable, so is $\mathbb{Q}$.

Theorem [Cantor]

$\mathbb{R}$ is uncountable.

The proof technique is called diagonalization.

Proof Idea It is sufficient to show that $(0,1)$ is uncountable.

Let $f: \mathbb{N} \xrightarrow[\text{onto}]{} (0,1)$ be a candidate enumeration of $\mathbb{R}$.

$$\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \frac{1/4}{1 - 1/2} = \frac{1/4}{2/4} = 1/2 = (0.10\ldots)_2$$

$$(0.0111111\ldots)_2 = (0.1000\ldots)_2$$

In decimal,

$$(0.999\ldots)_{10} = (1.000\ldots)_{10}$$

$$\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots = \frac{\frac{9}{10}}{1 - \frac{1}{10}} = \frac{9/10}{9/10} = 1.$$

Let $f : \mathbb{N} \to \{0,1\}^{\infty}$ be a candidate enumeration.

f						
0	$\boxed{0}$	1	0	1	0	. - - -
1	1	$\boxed{0}$	1	0	1	. - - .
2	.					
3	1	1	$\boxed{1}$	0	1	- - -
4	1	1	1	$\boxed{1}$	1	- - -
$\vdots$						

Consider

$0 \cdot 1100 \cdots$

There is no preimage under f.

Theorem [Cantor]

For any set A, there is no $f: A \xrightarrow[\text{onto}]{} \mathcal{P}(A)$