CS687 2023 HW2

Due: April 21, 2024

1. A probability measure μ on Σ^* is defined by the following rules.

   1. $\mu(\lambda) = 1$ and
   2. For every string w, $\mu(w) = \mu(w0) + \mu(w1)$.

   (The way to understand this is that $\mu(w)$ is the probability of all infinite sequences with w as a prefix. Then the second rule is finite additivity over disjoint unions.)

   Note the correction.

   If μ and ν are probabilities on Σ^*, such that $\nu(w) \neq 0$ for any string w, then show that μ/ν is a ν-martingale.

   A ν martingale is a function $m : \Sigma^* \to [0, \infty)$ such that $m(\lambda) \leq 1$ and for any string $w \in \Sigma^*$, we have

   $$m(w0)\nu(w0) + m(w1)\nu(w1) = m(w)\nu(w).$$

   [10]

2. Prove or disprove: if $d : \Sigma^* \to [0, \infty)$ is a martingale which does not assign 0 to any string, then d is the ratio of two probabilities on strings. **[10]**

3. Show that if there is a lower semicomputable martingale $m : \Sigma^* \to [0, \infty)$ such that it succeeds on $X \in \Sigma^\infty$- *i.e.*,

   $$\limsup_{n \to \infty} m(X[0 \dots n-1]) = \infty,$$

   then there is another lower semicomputable martingale $m' : \Sigma^* \to \infty$ such that

   $$\liminf_{n \to \infty} m(X[0 \dots (n-1)] = \infty.$$

   Hint: Use the "savings account" idea discussed in class. m' simulates m and every time m doubles its capital, m' adds an amount to the savings account and only uses the remaining capital to bet. **[10]**