CS687 2023 HW1

Due: February 17, 2024

Notation. $M(i)\downarrow$ means that the Turing machine M halts on input i.

1. Prove that every infinite computably enumerable language contains an infinite decidable subset. [10]

2. Let x be an arbitrary finite binary string. Do all permutations of x have the same plain Kolmogorov complexity of x? If so, prove your claim. Otherwise, construct a counterexample, and prove that your string has some permutation which has a significantly different Kolmogorov complexity. [10]

3. Let χ_H be the infinite binary sequence defined as follows. For any number i, the i^{th} bit of χ_H is 1 if the i^{th} Turing machine halts, and 0 otherwise. Since the Halting problem is uncomputable, χ_H is uncomputable. However, show that prefixes of χ_H are compressible: for all sufficiently large n, show that $C(\chi_H[0\dots n-1]) \leq \log n + O(1)$, where $\chi[0\dots n-1]$ denotes the n-length prefix of χ_H. (This shows that even when a string is uncomputable, it may be highly compressible.) [10]

4. Let ω be an infinite binary sequence defined by

$$\omega = \sum_{\substack{i\in\mathbb{N},\\ M(i)\downarrow}} \frac{1}{2^i}.$$

   Show that prefixes of ω are incompressible: for all sufficiently large n, $C(\omega[0\dots n-1]) \geq n - O(1)$. (The purpose of this question is to show that there are uncomputable strings which are also incompressible, in contrast to the previous question.) [10]

5. Let $f : \Sigma^* \to \Sigma^*$ be a total computable function. Then show that for every string x, we have $C(f(x)) \leq C(x) + O(1)$. (The purpose of this question is to show that you can never substantially increase the information in any string through computation: you can only destroy the information.) [10]

6. Show, using an argument using plain Kolmogorov complexity, that for all sufficiently large n, most strings of length n will <u>not</u> have a contiguous stretch of more than $\log^2 n$ zeroes anywhere in the string. (This shows how you can prove that some event happens with very high probability, using the theory of plain Kolmogorov complexity.) [10]

7. Show that $C(xy) \not\leq C(x) + C(y) + O(1)$. Also explain why it is not possible to have a prefix encoding e such that there is a positive constant c such that for all x, $|e(x)| \leq |x| + c$. [10]