Martingales

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1 Introduction

Let Σ be the binary alphabet, Σ^* the set of finite binary strings, and Σ^∞ be the set of infinite sequences.

Definition 1.1. The *cylinder* corresponding to a finite string w is the set of infinite binary sequences with w as a prefix defined by

$$C_w = \{\sigma \in \Sigma^\infty | w \text{ is a prefix of } \sigma\}.$$

Using the notion of cylinder sets, we can define the probability measure on Σ^∞, the set of *infinite* binary sequences. We will define the probability of cylinder sets.

Definition 1.2. A probability measure $\mu : \Sigma^* \to \infty$ on Σ^∞ is defined to be a function such that the following conditions hold.

$$\nu(\lambda) = 1$$
$$\nu(w) = \nu(w0) + \nu(w1), \qquad w \in \Sigma^*$$

The intent of the above definition is that for any string w, $\nu(w)$ is the measure of the cylinder C_w.

Example 1. For example, the function $\mu : \Sigma^* \to \infty$ defined by $\mu(w) = 2^{-|w|}$ satisfies the above conditions. This is called the *uniform probability measure*.

In the following discussion, we restrict ourselves to μ, the uniform probability measure on Σ^∞.

Definition 1.3. A *μ-martingale* is a function $d : \Sigma^* \to [0, \infty)$ such that

$$d(\lambda) = 1$$
$$d(w) = \frac{d(w0) + d(w1)}{2}.$$

We say that a μ-martingale d *succeeds* on an infinite sequence $\sigma \in \Sigma^\infty$ if

$$\limsup_{n\to\infty} d(\sigma[0 \ddots n-1]) = \infty.$$

The *success set* of a μ-martingale d is

$$S^\infty[d] = \left\{\sigma \in \Sigma^\infty \mid \limsup_{n\to\infty} d(\sigma[0 \ddots n-1] = \infty.\right\}.$$

We refer to μ-martingales as simply martingales. Intuitively, a martingale denotes a fair betting condition. We view $d(w)$ as the capital in our hand after w has occurred as the set of outcomes. The martingale condition states that the average value after the next outcome is equal to the present capital. We now look at a few examples.

Example 2. Consider $d : \Sigma^* \to [0, \infty)$ defined by $d(w) = 1$ for all strings w. Clearly, it satisfies the conditions for a martingale. On every infinite sequence σ, we know that $\limsup_{n\to\infty} d(\sigma[0 \ddots n-1]) = 1 < \infty$, so this simple martingale does not win on any sequence.

Example 3. Consider the following martingale $d : \Sigma^* \to [0, infty)$ defined by $d(\lambda) = 1$, and for any string w, $d(w0) = 2d(w)$, and $d(w1) = 0$. Then $\limsup_{n\to\infty} d(0^\infty[0 \ddots n-1]) = \lim_{n\to infty} 2^n = \infty$. This is a martingale that succeeds on the singleton set $\{0^\infty\}$. We can see that if there is any 1 in the infinite sequence, d attains 0 on that sequence, hence losing on it.

Example 3. Suppose we want to design a martingale that succeeds on all binary sequences whose asymptotic frequency of 0s is 3/4. Consider the following martingale, $d : \Sigma^* \to [0, \infty)$ defined by

$$d(\lambda) = 1$$
$$d(wb) = \begin{cases} \frac{3}{2}d(w) & \text{if } b = 0 \\ \frac{1}{2}d(w) & \text{otherwise.} \end{cases}$$

We can verify that it is a martingale.

We now show that d succeeds on any infinite binary sequence σ asymptotically has 3/4 fraction of zeroes. Let $A(\sigma, n)$ denote the number of zeroes in the first n bits of σ. Then for any $\varepsilon > 0$, for all sufficiently large n, we have

$$\left| \frac{A(\sigma, n)}{n} - \frac{3}{4} \right| < \varepsilon.$$

Then the value of d for sufficiently large n is

$$d(\sigma) \geq \left(\frac{3}{2}\right)^{\frac{3}{4}n - \varepsilon n} \left(\frac{1}{2}\right)^{\frac{1}{4}n + \varepsilon n} = \left(\frac{3^{3/4-\varepsilon}}{2}\right)^n,$$

For sufficiently small ε (e.g. $\varepsilon = 0.025$), we have $3^{3-4\varepsilon} > 2^4$, and hence the above expression grows exponentially in n.

It follows that

$$\limsup_{n\to\infty} d(\sigma[0 \ddots n-1]) = \infty,$$

thus succeeding on σ. (End of Example 3.)

2 Constructive martingales, and random sequences

In the classical mathematical setting, we know that sets with probability 0 can be characterized as the succcess set of martingales. Our goal now is to define constructive probability zero sets, and to show that there is a largest such set. By definition, there will be a martingale computable in a certain sense, which succeeds on every element in this set. So every element in this set is *non-random*

The complement of this set will be called the set of *Martin-Löf random sequences*.

Definition 2.1. A martingale $d : \Sigma^* \to [0, \infty)$ is called *constructive* or *lower semicomputable* if there is a total computable function $\hat{d} : \Sigma^* \times \mathbb{N} \to \mathbb{Q} \cap [0, \infty)$ such that the following hold.

1. (Monotonicity) $\forall w \in \Sigma^*$, $\forall n$, $\quad \hat{d}(w, n) \leq \hat{d}(w, n+1) \leq d(w)$.
2. (Convergence) $\forall w \in \Sigma^*$, $\limsup_{n\to\infty} \hat{d}(w, n) = d(w)$.

A set $A \subseteq \Sigma^\infty$ is called *constructive null* if there is a constructive martingale $d : \Sigma^* \to [0, \infty)$ such that $A \subseteq S^\infty[d]$ - *i.e.* d succeeds on every sequence in A.

Example 4. The martingale in Example 3 is constructive, with $\hat{d}(w, n) = d(w)$ for every w, n as the lower semicomputation. Hence, the set of all infinite binary sequences with asymptotically 3/4 fraction of zeroes, forms a constructive null set.

Theorem 2.2. *Let d_1, d_2, $\cdots : \Sigma^* \to [0, \infty)$ be a computable enumeration of constructive martingales. Then $d = \sum_{i=1}^{\infty} 2^{-i} d_i$ is a constructive martingale such that*

$$S^\infty[d] \supseteq \bigcup_{i=1}^{\infty} S^\infty[d_i].$$

Proof. It can be verified by direct calculation that $d(\lambda) = 1$ and $d(w) = (d(w0) + d(w1))/2$. Hence d is a martingale.

Consider the function $\hat{d} : \Sigma^* \times \mathbb{N} \to \mathbb{Q} \cap [0, \infty)$ defined by $\hat{d}(w, n) = \sum_{i=1}^{n} 2^{-i} \hat{d}_i(w, n)$, we can see that $\hat{d}$ is a lower semicomputation of d. Thus d is a constructive martingale.

Suppose $\sigma \in S^\infty[d_j]$ for some $j \in \mathbb{N}$. Then

$$\limsup_{n\to\infty} \sum_{i=1}^{\infty} 2^{-i} d_i(\sigma[0 \cdots n-1]) \geq 2^{-j} \limsup_{n\to\infty} d_j(\sigma[0 \cdots n-1]) = \infty,$$

hence $\sigma \in S^\infty[d]$. ☐

Note. For the above proof to work, it is necessary that $\hat{d}_i(w, n)$ is defined for all i, w, and n. Otherwise the convex combination is undefined.

We also show that there is a computable enumeration of all the constructive martingales. Together with the above theorem, this enables us to establish that there is a universal c.e. martingale.

Theorem 2.3. *There is a computable enumeration of constructive martingales.*

Proof. TBD. ☐