CS 687 Homework 1

1. What is the cardinality of the set of continuous functions from (0,1) to (0,1)? Prove your claim.

2. Show that the intersection of finitely many computably enumerable languages is computably enumerable.

3. Show that the computably enumerable union of computably enumerable languages is computably enumerable.

4. If $f : \mathbf{N} \to \mathbf{N}$ is a partial computable function, then show that the range of f, defined below, is Turing acceptable.
$$\{y \mid \exists x \in \mathbf{N} f(x) = y\}.$$

5. If $b_0 b_1 \ldots b_n$ is a finite string, then is it true that all its "circular left shifts" $b_i \ldots b_{n-1} b_0 \ldots b_{i-1}$, where $0 < i < n$ have the same complexity, up to an additive constant? Prove your claim.

6. (Data Processing Inequality) Suppose $\phi : \Sigma^* \to \Sigma^*$ is a total computable function. Then show that there is a constant, c_ϕ depending on ϕ, such that for all $x \in \Sigma^*$,
$$K(\phi(x)) \leq K(x) + c_\phi, \qquad C(\phi(x)) \leq C(x) + c_\phi.$$
This inequality shows that no computable process can increase the complexity of a string, even though it can decrease the complexity.

7. Consider finite prefixes of the characteristic sequence of the Halting problem, *i.e.* for all $n \in \mathbf{N}$,
$$h_n = b_0 b_1 \ldots b_{n-1}$$
where the $b_i = 1$ if $s_i \in H$, and $b_i = 0$ if $s_i \notin H$. Show that for all large n, $C(h_n) \leq \log_2 n + O(1)$. This shows that there are uncomputable languages which have very low complexity.

8. Show that for any injective function $f : \Sigma^* \to \Sigma^*$, the proportion of strings of length n which have images of length $n - c$ or shorter, is at most 2^{-c}.

9. Show that there is no prefix-free encoding $p : \Sigma^* \times \Sigma^* \to \Sigma^*$ such that for every x and y, $|p(x, y)| \leq |x| + |y| + O(1)$.