CS 744: Pseudorandomness Generators

Lecture Notes 19: Introduction to Expander Graphs

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We recall a few facts in linear algebra related to spectral graph theory. If $x = a+ib$ is a complex number, where a and b are reals, then we denote its complex conjugate by $x* = a - ib$.[1]

If M is an $n \times n$ complex matrix, $\lambda \in \mathbb{C}$, $v \in \mathbb{C}^n - \{0\}$ is a vector, such that $Mv = \lambda v$, then we say that λ is an *eigenvalue* and v is an *eigenvector* corresponding to the eigenvalue λ of M.

$Mv = \lambda v$ iff $(M - \lambda I)v = 0$ iff $det(M - \lambda I) = 0$. Since this determinant is a univariate (in λ) polynomial of degree n, it has at most n distinct roots. Hence an $n \times n$ matrix has precisely n eigenvalues, counting multiplicities.

We will be interested in the connection between undirected graphs and their adjacency matrices. If G is a graph with n vertices labelled 1 through n, then the $n \times n$ adjacency matrix has its $(i,j)^{\text{th}}$ element equal to 1 precisely when (i,j) is an edge in G.

0.1 Undirected Graphs

When G is undirected, (i,j) is an edge precisely when (j,i) is. Hence the adjacency matrix of an undirected graph is symmetric. The eigenvalues of a symmetric matrix are real, as we now show.

Definition 1. *A matrix $M \in \mathbb{C}^{n\times n}$ is* Hermetian *if $M_{ij} = M^*_{ji}$ for every i, j.*

Lemma 2. *If M is Hermetian, then all its eigenvalues are real.*

Proof. If $Mv = \lambda v$, then we need to show that $\lambda = \lambda^*$. Define the inner product over vectors in $\mathbb{C}^n$ by

$$\langle v, w \rangle = \sum_{i=1}^{n} v_i^* \cdot w_i.$$

[1]The material in this lecture is taken from Lecture 2 of Luca Trevisan's course CS359G: Expander Graphs and their Applications.

Now,

$$\langle Mx, x\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij} * x_j * \cdot x_i = \sum_{j=1}^{n}\sum_{i=1}^{n} M_{ji} x_i \cdot x_j^* = \langle x, Mx\rangle,$$

since M is Hermetian. Now, if x is an eigenvector corresponding to the eigenvalue λ, then $\langle Mx, x\rangle$ $= \langle \lambda x, x\rangle$. We have

$$\langle Mx, x\rangle = \langle \lambda x, x\rangle = \lambda^* \langle x, x\rangle = \langle x, Mx\rangle = \lambda\langle x, x\rangle$$

Since x is non-zero, it follows that λ is real. □

Definition 3. *If M is a real symmetric matrix and λ its eigenvalue (hence real), then λ admits a real eigenvector.* [2] *The set of real eigenvectors is a vector subspace of $\mathbb{R}^n$, called the* eigenspace

0.2 Some properties of eigenvalues and eigenvectors

We will look at the relation between the eigenvalues of the adjacency matrix of a graph and the combinatorial properties of the graph.

Lemma 4. *The eigenspaces of distinct eigenvalues of a real symmetric matrix M are orthogonal.*

Proof. Let λ be an eigenvalue with a real eigenvector x and λ' be an eigenvalue distinct from λ with a real eigenvector y. Then $\langle Mx, y\rangle$ is $\lambda\langle x, y\rangle$ and $\langle x, My\rangle$ is $\lambda'\langle x, y\rangle$. Since all the quantities involved in $\lambda Mx, y\rangle$ and $\langle x, My\rangle$ are real, we can verify that these two quantities are equal. Thus, $(\lambda - \lambda')\langle x, y\rangle$ is zero, hence $\langle x, y\rangle$ is zero. □

Definition 5. *The* algebraic multiplicity *of an eigenvalue λ of M is the multiplicity of λ as a root of M.*

Its geometric multiplicity *is the dimension of its eigenspace.*

For a real symmetric matrix M, the algebraic and geometric multiplicities coincide for every eigenvalue. Thus if $\lambda_1, \ldots, \lambda_n$ are its eigenvalues, then M has an orthonormal basis of real eigenvectors $v_1, \ldots, v_n$.

Now we build connections between spectral and combinatorial properties of graphs.

Lemma 6. *Let M be a symmetric (not necessarily real) matrix. Then its largest eigenvalue λ_1 is*

$$\lambda_1 = \sup_{x\in\mathbb{R}^n,\ ||x||=1} x^T M x. \tag{1}$$

Proof. Since $\{x \in \mathbb{R}^n,\ ||x|| = 1\}$ is a compact space, there is a vector $y \in \mathbb{R}^n$ which attains the supremum. We will show that the supremum in (**??**) is upper bounded and lower bounded by λ_1, which yields the conclusion.

To show that the supremum in (**??**) is lower bounded by λ_1, we note that v_1, the eigenbasisvector corresponding to λ_1 satisfies

$$v_1^T M v_1 = v_1^T(\lambda_1 v_1) = \lambda_1 v_1^T v_1 = \lambda_1. \tag{2}$$

Since there is one vector in the compact space which attains value λ_1, it follows that λ_1 bounds the supremum in (**??**) from below.

To establish the upper bound, expand y in the basis as

$$y = \alpha_1 v_1 + \ldots \alpha_n v_n, \tag{3}$$

whence the supremum is at least $y^T M y = \sum_{i=1}^n \alpha_i^2 \lambda_i$. Since $\sum_{i=1}^n \alpha_i^2 = 1$ and $\lambda_i \leq \lambda_1$ for $1 \leq i \leq n$, we have that the supremum is at most λ_1. □

0.3 Undirected d-regular graphs