Martingales and Restricted Ratio Betting

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August 5, 2019

Abstract

Let A and B be two finite sets of computable real numbers which denote the allowable *wagers*, *i.e.* the additive difference of capital at any betting position, . Following the terminology in Chalcraft et. al., an A-martingale is a martingale whose wagers are limited to elements in A, and a B-martingale has wagers limited to elements in B. Extending the work of Chalcraft et. al., Bavly and Peretz establish necessary and sufficient conditions for some A-martingale to succeed betting on sequences that B-martingales can succeed betting on for arbitrary infinite sets of reals.

In this paper, we investigate the analogous question of comparative betting power of martingales when the *ratios* of bets are restricted to a finite set of rationals which excludes 1. This contrasts with the setting of simple martingales and almost simple martingales as investigated by Ambos-Spies, Mayordomo, Wang and Zheng. We derive necessary and sufficient conditions for deciding when a set of ratios allows greater power in betting as compared to another. Analogous to a recent work of Teutsch, we establish that success and strong success of martingales are distinct notions in the setting of restricted ratio betting.

Keywords: Martingales, Computability.

1 Introduction

Let A and B be two finite sets of computable real numbers which denote the allowable *wagers* that martingales can make. Following the terminology in Chalcraft et. al. [5], an A-martingale is a martingale whose wagers are limited to elements in A, and a B-martingale has wagers limited to elements in B. In Chalcraft et. al. [5], the authors establish necessary and sufficient conditions for some A-martingale to succeed betting on sequences that B-martingales can succeed betting on. Integer-valued martingales, where the wagers are guaranteed to be integers, and martingales with restricted wagers have been objects of recent studies by Bienvenu, Stephan and Teutsch [4], [9], and Peretz [8], culminating in the elegant work of Bavly and Peretz [3] giving a full characterization of when a martingale can win over more sequences with one set of real-valued wagers than with another possibly infinite set of real-valued wagers.

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In this paper, we investigate the analogous question of comparative betting power of martingales when the *ratios* of bets are restricted to a finite set. Without loss of generality, we will restrict the ratios to rational numbers, and the general case of finite sets of computable real ratios is similar.

A very similar setting has been explored in the context of *simple* martingales and *almost simple* martingales, in the work of Ambos-Spies, Mayordomo, Wang and Zheng [1]. A simple martingale is one that can bet either q, $2-q$ or 1 times its current capital on the next bit, and an almost simple martingale is one that is allowed to bet from a finite set of ratios which includes 1. They show that regardless of which finite ratios are used, the set of randoms on which the martingales fail is exactly the same. Moreover, this class is precisely the set of *Church-Stochastic* sequences.

However, we show in this work that the situation is radically different when we make the ostensibly minor change that martingales are allowed to bet from a finite set of rational betting ratios, but they are never allowed to bet evenly. This results in behavior which depends on the actual set of ratios used. We show a necessary and sufficient condition for a set of randoms with respect to one set of ratios to be a proper subset of the randoms for another set of ratios, as long as neither contain 1. We say that the ratio set with the larger success set *dominates* the other. This is the first setting where restricting the ratio set leads to a separation of different degrees of randomness. In particular, we show the following (definitions are provided in the next section).

Theorem. Let A and B be finite sets of ratios in the unit interval. Then the set of random sequences with respect to computable A-martingales is a strict subset of the set of random sequences with respect to computable B-martingales succeed if and only if $\max(B) < \max(A)$.

We also investigate the dual notions of *success* where the capital of the martingale infinitely often exceeds any given bound, and that of *strong success* where the capital of the martingale almost always stays above any given bound. In the context of computable martingales, we know that for every sequence on which some c.e. martingale succeeds, there is some other c.e. martingale which strongly succeeds, employing what we call the *savings account method* (see for example, [6]). However, Teutsch has showed that in the context of martingales with integer wagers, these two notions lead to distinct notions of randomness [9], even though for rational wagers, they are equivalent. We show that these two notions are distinct for valid ratio sets as well, giving more insight into the power and limitations of the *savings account method*.

2 Preliminaries and Notation

We denote the set of natural numbers by $\mathbb{N}$, rationals by $\mathbb{Q}$ and reals by $\mathbb{R}$. For any set of numbers S, the notation S^+ denotes the set of positive numbers in S. For functions $f, g : \mathbb{N} \to \mathbb{N}$, we say that f is $o(g)$ if $\lim_{n\to\infty} f(n)/g(n) = 0$, and $f = \Theta(g)$ if there are positive constants c and C such that for all large enough n, $cg(n) \leq f(n) \leq Cg(n)$.

Let Σ denote the binary alphabet. We denote finite binary strings by Σ^* and infinite binary sequences by Σ^ω. For positions $m < n$, finite strings w and infinite sequences X, $w[m \dots n]$ and $X[m \dots n]$ are the respective substrings consisting of all characters from the m^{th} and n^{th} position (both ends inclusive). For brevity, we write $X \upharpoonright n$ for $X[0 \dots n]$.

Definition 1. A *martingale* $M : \Sigma^* \to [0, \infty)$ is a function such that the following hold.

1. $M(\lambda) = 1$.

2. For every string w, $M(w) = M(w0) + M(w1)$.

We can interpret such a function as the capital of a player betting on the outcomes of a betting game on the bits of an infinite sequence. If we interpret $M(w)$ as the capital at the string w, then martingale represents a *fair betting* scenario where the expected capital after a bet is the same as the capital before the bet.

Definition 2. A martingale $M : \Sigma^* \to [0, \infty)$ is said to *succeed* on a sequence X if

$$\limsup_{n\to\infty} M(X \upharpoonright n) = \infty, \tag{1}$$

and is said to *strongly succeed* on X if

$$\liminf_{n\to\infty} M(X \upharpoonright n) = \infty. \tag{2}$$

The *success set* of the martingale M is defined as

$$S^\infty[M] = \{X \mid M \text{ succeeds on } X\} \tag{3}$$

and the *strong success set* of the martingale is defined as

$$S^\infty_{\text{str}}[M] = \{X \mid M \text{ strongly succeeds on } X\}. \tag{4}$$

Sequences in $\Sigma^\omega - S^\infty[M]$ are said to be random with respect to M.

A real number r is said to be *computable* if there is a computable function $\hat{r} : \mathbb{N} \to \mathbb{Q}$ such that for every n, $|\hat{r}(n) - r| < 2^{-n}$. A martingale $M : \Sigma^* \to [0, \infty)$ is said to be *computable* if $M(w)$ is a computable real number, uniformly in w.

Definition 3. A finite set A of computable real numbers in $(0, 1)$ is called a *ratio set*. The *closure* of A, denoted $\overline{A}$ is $A \cup \{s \mid 2 - s \in A\}$.

For a ratio set A, an A-*martingale* is a martingale M such that for every string w and bit b, we have $M(wb)/M(w) \in \overline{A}$.

Note that this notion allows us to retain $M(\lambda) = 1$. We shall usually be concerned with computable A-martingales; if A is finite then this is equivalent to saying that $(w, b) \mapsto M(wb)/M(w)$ is a total computable function.

Extending the notion of success sets of a martingale, we introduce the following.

Definition 4. For a ratio set A, the *success set* of A is defined by

$$S^\infty[A] = \{X \mid \exists \text{ a computable } A\text{-martingale } M \text{ such that } X \in S^\infty[M]\} \tag{5}$$

and the *strong success* set is defined by

$$S^\infty[A] = \{X \mid \exists \text{ a computable } A\text{-martingale } M \text{ such that } X \in S^\infty_{\text{str}}[M]\}. \tag{6}$$

The set of sequences $\Sigma^\omega - S^\infty[A]$ are called A-*randoms*.

3 Ratio Sets versus Restricted Wagers

Given a martingale $M : \Sigma^* \to [0, \infty)$, a *wager* at a string w on the bit b is the difference $M(wb) - M(w)$. In Chalcraft et al.[5], the authors observe that if there is a computable real k such that the wagers in B is a subset of the wagers in kA, then for every sequence X on which some B-martingale succeeds, there is an A-martingale which succeeds on X as well.

Given two finite sets of betting ratios, we want to find necessary and sufficient conditions for the success set of a B-martingale to be subsumed in the success set of some A-martingale. We begin by arguing that the methods for dealing with finite sets of wagers fail when we consider finite sets of betting ratios.

We first show that $A \subseteq kB$ may be insufficient to guarantee that $S^\infty[B] \subseteq S^\infty[A]$.

Let M be a B-martingale. Then the strategy in Chalcraft et. al. for the additive wagers is to bet k times the amount that M bets at each position. However, if we try to imitate this strategy for restricted ratios using an A martingale which bets k times what M bets, it is easy to check that does not define a martingale, since

$$N(w0) + N(w1) = N(w)k\frac{M(w0) + M(w1)}{M(w)} = 2kN(w),$$

and this is equal to $N(w)$ only when $k = 1$. Hence even if $B \subseteq kA$ is a sufficient condition, the construction may not be a straightforward adaptation of the technique in Chalcraft et. al. [5].

We now derive the necessary and sufficient conditions on the set of betting ratio sets A and B to ensure that $S^\infty[B] \subsetneq S^\infty[A]$.

4 Unique betting ratios against other unique ratios - Sufficient conditions for domination

In this section, we examine the simplest scenario, namely, the one where the sets of ratios A and B are both singletons (since neither can be $\{1\}$, their respective closures contain exactly two elements). We establish the sufficient condition for $S^\infty[B] \subsetneq S^\infty[A]$ that the ratio in A is greater than the ratio in B.

Observe that for any $x \in (0, 1)$, we have $g(x) = x(2-x) < 1$. Moreover, g is monotone increasing in $(0, 1)$. We can refine this observation, as in the following lemma. We utilize a *threshold function* in the proof, which we first define.

Definition 5. Let $x \in (0, 1)$. Then the *threshold function of* x, $h_x : \mathbb{N} \times \mathbb{R} \to \mathbb{N}$ is defined by

$$h_x(j, r) = \min\{i \mid x^j(2-x)^{j+i} > r\}. \tag{7}$$

To show that h_x is well-defined and total, we observe by taking logarithms that $x^j(2-x)^{i+j}$ is greater r precisely when

$$i > -j\left(1 + \frac{\log x}{\log(2-x)}\right) + \frac{\log r}{\log(2-x)}.$$

This can always be achieved by making i large enough, hence there must be a minimal such i. In particular, if $r > 1$ then $h_x(j, r) > 0$.

Lemma 6. *Suppose $a > b$. There is a positive constant c such that for sufficiently large $r > 0$, and every positive i, j, if $b^j(2-b)^{i+j} > r$, then $a^j(2-a)^{i+j} > r^c$.* [1]

[1] r need not be greater than 1, but i has to be positive.

Proof. By assumption, we know that

$$i > -j\left[1+\frac{\log b}{\log(2-b)}\right]+\frac{\log r}{\log(2-b)}.$$

Fix $c = \log(2-a)/\log(2-b) > 0$. It suffices to show that

$$\begin{aligned}-j\left[1+\frac{\log b}{\log(2-b)}\right]+\frac{\log r}{\log(2-b)} &> -j\left[1+\frac{\log a}{\log(2-a)}\right]+\frac{c\log r}{\log(2-a)}\\ &= -j\left[1+\frac{\log a}{\log(2-a)}\right]+\frac{\log r}{\log(2-b)},\end{aligned}$$

since the above implies that $a^j(2-a)^{i+j} > r^c$. Since $\log x/\log(2-x)$ is monotone increasing in $(0,1)$, the inequality above follows easily. □

Lemma 7. *Suppose $A = \{a\}$ and $B = \{b\}$, where $0 < b < a < 1$. Then $S^\infty[B] \subseteq S^\infty[A]$.*

Proof. Let N be a computable B-martingale which succeeds on X, and r be a number sufficiently large so that $r^c > 1$, where $c = \log_{(2-b)}(2-a) > 0$. Then for every m, there is some $n > m$ such that $N(X \restriction n)/N(X \restriction m) > r$. Equivalently,

$$\frac{N(X \restriction n)}{N(X \restriction m)} = b^j(2-b)^{i+j}, \text{ where } i > f_b(j,r).$$

Then the A-martingale M which bets $(2-a)$ of its current capital when N bets $(2-b)$ on a bit, and a when N bets b, satisfies

$$\frac{M(X \restriction n)}{M(X \restriction m)} = a^j(2-a)^{i+j} > r^c > 1,$$

by Lemma 6. Since N is computable and division over computable non-zero reals is computable, it follows that M is computable as well. Since c does not depend on r, hence, $X \in S^\infty[M]$. □

We now show that A strictly dominates B in the sense that there are sequences on which some computable A-martingale succeeds, but no computable B-martingale does. We construct such a sequence X by a finite extension, permitting some specific A-martingale to succeed on X, but defeating all B-martingales. The proof relies on a combinatorial estimate which shows that at any stage in the construction, we can always extend our current finite prefix of X in a manner so as to defeat all B-martingales.

Lemma 8. *For $x \in (0,1)$, let $c_x = \frac{1}{1-\log_{(2-x)}(x)}$. If $0 < b < a < 1$, then we have $0 < c_b < c_a < 0.5$.*

Proof. This follows from the observation that if $x \in (0,1)$, then $-\infty < \log_{(2-x)}(x) < -1$, hence $0 < c_x < 0.5$. Since $\log_{(2-x)}(x)$ is monotone increasing for $x \in (0,1)$, it follows that for $0 < b < a < 1$, we have $c_b < c_a$. □

Theorem 9. *Let $0 < b < a < 1$, $A = \{a\}$ and $B = \{b\}$. Then $S^\infty[A] - S^\infty[B]$ is non-empty.*

Proof. Fix $N_0 : \Sigma^{<\omega} \to [0,\infty)$ by:

$$\begin{aligned} N_0(\lambda) &= 1,\\ N_0(\sigma b) &= \begin{cases}(2-a)N_0(\sigma), & \text{if } \sigma \in \Sigma^{<\omega} \text{ and } b = 0,\\ aN_0(\sigma), & \text{if } \sigma \in \Sigma^{<\omega} \text{ and } b = 1.\end{cases}\end{aligned}$$

That is, N_0 is an A-martingale which always bets on 0. Let N_i, $i \geq 1$, be an enumeration of all computable B-martingales. Notice that our construction does not have to be effective; we merely need to show the existence of some $X \in S^\infty[A] - S^\infty[B]$.

We construct a non-empty set of sequences in $S^\infty[N_0] - \cup_{i\geq 1} S^\infty[N_i]$. For convenience, we assume that the initial capital of the i^{th} martingale, $N_i(\lambda) < (2-b)^{-L_i}$, where L_i is the length of the i^{th} stage in the construction that follows.

The construction procceds in stages. At stage k, we determine prefixes τ such that $N_0(\tau) \geq (2-a)^k$, but for every $1 \leq i \leq k$, there is a constant c_i depending only on i such that

$$\max\{N_i(\tau \upharpoonright m) \mid 0 \leq m \leq |\tau|\} \leq c_i,$$

i.e. N_i does not exceed c_i on any prefix of τ.

Each stage is divided into two phases. During the first phase, we construct strings on which every one of the finite number of martingales currently under consideration have roughly equal number of gains and losses. Subsequently, we extend them with sufficient number of zeroes in phase II to allow N_0 to attain the required goal while restricting the gains of the B-martingales.

For the construction, we require a few preliminary results from probability theory, which we now describe.

Define the random variables $f_{i,m} : \Sigma^\omega \to \{0,1\}$ for i, $m \in \mathbb{N}$ by

$$f_{i,m}(X) = \begin{cases} 0 & \text{if } N_i(X \upharpoonright m) > N_i(X \upharpoonright m-1) \\ 1 & \text{otherwise.} \end{cases}$$

For any martingale N_i, we can verify that $f_{i,1}$, $f_{i,2}$, $\ldots$, forms a sequence of independent, identically distributed random variables, as follows. Since the martingales cannot bet evenly, for any finite subcollection f_{i,m_0}, f_{i,m_1}, $\ldots$, $f_{i,m_{j-1}}$ of random variables, the probability that $f_{i,m_0} = b_0$, $\ldots$, $f_{i,m_{j-1}} = b_{j-1}$, for any sequence $(b_0, \ldots, b_{j-1})$ of bits is clearly $\frac{1}{2^j}$. This shows that the sequence of random variables is independent. Since the probability that any particular $f_{i,j} = 0$ is 0 is 0.5, they are also identically distributed. The average $E[f_{i,0}]$ is thus 0.5.

The Strong Law of Large Numbers [7] implies that, for any $\varepsilon > 0$, we have that

$$\mu\left(\{X \in \Sigma^\omega \mid \forall^\infty n \ |f_{i,n}(X) - 0.5| < \varepsilon\}\right) = 1, \tag{8}$$

where μ is the Lebesgue measure. In the above equation, the least n after which $f_{i,n}(X)$ stays within ε of 0.5 can vary, depending on X.

By Egorov's theorem [2], there is a set with positive measure where this non-uniform almost everywhere convergence in (8) can be made uniform - *i.e.* for every $\varepsilon > 0$, there is an N such that the set

$$\{X \in \Sigma^\omega \mid \forall n \geq N |f_{i,n}(X) - 0.5| < \varepsilon\}$$

has positive measure.

We fix $\varepsilon \in (c_b, c_a)$ where c_b and c_a are defined as in Corollary 8.

Stage k=1. Let ℓ_0 be a length such that

$$S_0^0 = \{X \in \Sigma^\omega \mid \forall n \geq \ell_0 \ |f_{0,n}(X) - 0.5| < \varepsilon\}$$

has positive probability. Similarly, let $\ell_1 > \ell_0$ be a length such that the set

$$S_1^0 = \{X \in S_0^0 \mid \forall n \geq \ell_1 \ |f_{1,n}(X) - 0.5| < \varepsilon\}$$

has positive probability. By definition $S_1^0 \subseteq S_0^0$.

Let T be the set of prefixes of length ℓ_1 of elements in S_1^0. For all elements of T, both N_0 and N_1 have roughly equal number of gains and losses. This ends Phase I.

For every $\tau \in T$, N_0 has at least $\ell_1(0.5-\varepsilon)$ gaining positions, and N_1 has at most $\ell_1(0.5+\varepsilon)$ gaining positions. Append $\ell_1(0.5+\varepsilon-c_a)$ many zeroes to τ. Then N_0 has at least $\ell_1(1-c_a)$ gains on this extension, but N_1 has strictly less than $\ell_1(1-c_b)$ gains. Hence N_0 attains 1 on this extension, but N_1 does not.

Let

$$T_1 = \{\tau 0^{\ell_1(0.5+\varepsilon-c_a)} \mid \tau \in T\}.$$

Designate the lengths of the strings in T_1 by L_1. The ends stage 1.

Stage $k > 1$. Inductively, assume that we have defined a set T_{k-1} of strings where N_0 has attained $(2-a)^{k-1}$, but for $1 \leq i \leq k-1$, the capital gained by N_i anywhere along the prefixes of strings in T_{k-1} is at most

$$c_i = (2-b)^{\sum_{j=1}^{i} L_i}.$$

Note that this upper bound on the capital attained by N_i depends only on i, and does not depend on the stage number. Further, we assume that for every $\tau \in T_{k-1}$ and all $1 \leq i \leq k-1$, $N_i(\tau) \leq 1$ - *i.e.* every B-martingale considered in stage $k-1$ ends with capital at most 1.

In the k^{th} stage, we extend strings in T_{k-1} to build a set T_k. Each $\sigma \in T_k$ satisfies the following conditions.

1. $N_0(\sigma) \geq (2-a)^k$.
2. For every ρ such that $\tau \prec \rho \preceq \sigma$, where $\tau \in T_{k-1}$ and $\sigma \in T_k$, and for every $1 \leq i \leq k-1$, we have $N_i(\rho) \leq c_i$ - *i.e.*, during the stage k, N_1, $\dots$, N_{k-1} do not exceed capital c_1, $\dots$, c_{k-1} respectively.
3. $N_k(\sigma) \leq c_k$.

Let $[T_{k-1}]$ denote the set of all infinite length extensions of strings in T_{k-1}. Let ℓ_0 be a length such that the set

$$S_0^k = \{X \in [T_{k-1}] \mid \forall n \geq \ell_0 \ |f_{0,n}(X) - 0.5| < \varepsilon\}$$

has positive probability. Inductively, for $1 \leq i \leq k$, let $\ell_i > \ell_{i-1}$ be a length such that the set

$$S_i^k = \{X \in S_{i-1}^k \mid \forall n \geq \ell_0 \ |f_{i,n}(X) - 0.5| < \varepsilon\}$$

has positive probability. Let T be the set of ℓ_k-long prefixes of the sequences in S_k^k. This ends Phase I of stage k.

Extend each member of T with $0^{\ell_k(0.5+0.5\varepsilon-c_a)+k}$. Call the resulting set of strings T_k, and their length, L_k. This ends Phase II of stage k.

By construction, for every $\sigma \in T_k$, $N_0(\sigma) \geq (2-a)^k$.

For $1 \leq i \leq k-1$, we show that N_1 does not gain greater capital in stage k over its maximum gain in stage $k-1$. The crucial property we maintain during the construction is that for any martingale N_i, $1 \leq i \leq k-1$, the maximum length on which it can succeed during phase I remains the same in every stage after i.

For the purpose of the following discussion, let $\tau \in T_{k-1}$ and $\sigma \in T_k$. We use the letter ρ to designate an extension of a string in T_k which is a prefix of some string in T_k.

Clearly, $N_1(\tau) < 1$ for every $\tau \in T_{k-1}$. It has at most $(2-a)^{\frac{\ell_1}{2}} \leq c_1$ on any extension ρ in stage k. For any ρ having length $\geq \ell_1 + L_{k-1}$, the number of gains that N_1 has in stage k, is at most $|\rho|(0.5+\varepsilon)$ in Phase I, and is at most $|\rho|(1-c_b)$ in Phase II, hence $N_1(\rho) < 1$.

Similarly, for any N_i, $2 \leq i \leq k-1$, for any $\rho \prec \tau$, $\tau \in T_{k-1}$, such that $|\rho| > \ell_i + L_{k-1}$, we know that N_i has at most $|\rho|(0.5+\varepsilon)$ gains on ρ, hence $N_i(\rho) < N_i(\tau)$. Thus the maximum value that N_i attains in Phase I is $N_i(\tau)(2-b)^{\sum_{j=1}^{i} \ell_j}$, which is less than c_i. For any ρ longer than $\ell_i + L_{k-1}$, then number of gains that N_i has in stage k does not exceed $|\rho|(0.5+\varepsilon)$ in Phase I, and does not exceed $\rho(1-c_b)$ in Phase II, hence $N_i(\rho) < 1$.

For the k^{th} martingale, observe that $N_k(\tau)$ is at most 1, since $|\tau| < L_{k-1}$, and we have set the initial capital of N_k to ensure that $N_k(\tau) < 2^{-L_k} 2^{L_{k-1}} < 1$. As above, for any extension ρ of $\tau \in T_{k-1}$, $\ell_k + L_{k-1} < |\rho| \leq L_k$, the construction ensures that $N_k(\rho) < 1$.

□

5 Strong success versus success

In the setting of computably enumerable, computable, or resource-bounded martingales, it is known [6] that the set of sequences on which martingales strongly succeed is precisely the set of sequences on which they succeed. The proof of this fact employs the "savings account trick", described roughly as follows. Consider a martingale M and a sequence X on which it succeeds. We construct another martingale N which, whenever M doubles its money, transfers 1 dollar to its "savings account" and bets using only its remaining amount. It is easy to see that N strongly succeeds on X. Recently, Teutsch [9] has shown that interestingly, when wagers are restricted to integers, the notion of strong success is different from success - the savings account trick does not work in this setting. Teutsch also shows that for rational wagers, these notions coincide.

We now show that in restricted ratio betting with rational ratios, the notion of strong success is different from success. It is difficult to adapt the savings account trick to our setting. When we transfer the capital to the savings account, it is not clear how to define the betting ratios followed by the new martingale. It is not obvious that the betting ratios allowed for N are identical to that allowed for M.

Indeed, we show that strong success and success are distinct notions in our current setting.

Lemma 10. *Let A be a valid ratio set. Then there is a sequence $X \in S^{\infty}[A] - S^{\infty}_{str}[A]$.*

Proof. Let $\max(A) = a < 1$. Consider an enumeration M_1, M_2, ... of computable A-martingales. Without loss of generality, let M_1 be the martingale which bets $2-a$ of its current capital on 0, at every position. We construct a sequence X on which M_1 succeeds, but on which none of the above martingales, including M_1, succeed strongly. The construction proceeds by fixing finite prefixes of X, in stages.

At stage $s = 0$, let $\sigma_0 = \lambda$ be the prefix of X.

At every stage $s \geq 1$, we set a prefix σ_s of X to meet the following positive requirement $P_{1,s}$ and negative requirements $R_{i,s}$ where $1 \leq i \leq s$.

- $P_{1,s}$: There is a string γ, $\sigma_{s-1} \preceq \gamma \preceq \sigma_s$, where $M_1(\gamma) \geq s$.
- $R_{i,s}$: There is a string τ, $\sigma_{s-1} \preceq \tau \preceq \sigma_s$, where $M_i(\tau) \leq 1/2$.

During the stage s, we maintain a candidate extension $\hat{\sigma}_s \succ \sigma_{s-1}$. The string σ_s is an extension of this candidate.

Construction

At stage s, $s \geq 1$, we initialize $\hat{\sigma}_s$ to σ_{s-1}.

In substage 1 of stage s, we meet $P_{1,s}$ and $R_{1,s}$.

We know that M_1 bets on all paths extending $\hat{\sigma}_s$. In particular, there is a $\gamma \succeq \hat{\sigma}_{s-1}$ on which $M_1(\gamma 0) > M_1(\gamma 1)$. Take the lexicographically least such γ. We set the new value of $\hat{\sigma}_s$ to $\gamma 1$, and repeat the process until $M_1(\hat{\sigma}_s) < 1/2$. This process takes at most

$$\log_a \frac{1}{2M_1(\sigma_{s-1})}$$

such losing branches.

This meets the requirement $R_{1,s}$.

To meet $P_{1,s}$, in a similar manner, take $\log_{(2-a)} 2s$ winning branches for M_1 to obtain a string γ extending $\hat{\sigma}_s$ to ensure $M_1(\gamma) \geq s$. Set the new value of $\hat{\sigma}_s$ to γ.

Now, at each substage $2 \leq k \leq s$, of stage s, extend the current prefix of X to a new prefix where the requirement $R_{k,s}$ can be met as follows.

Note that M_k bets on a dense set of paths extending $\hat{\sigma}_s$. Then there is a $\gamma \succ \sigma_{s-1}$ on which $M_k(\gamma 0) \neq M_k(\gamma 1)$. If $M_k(\gamma 0) > M_k(\gamma 1)$, then we set the new value of $\hat{\sigma}_s$ to $\gamma 1$, else to $\gamma 0$. After at most

$$\log_a \frac{1}{2M_k(\sigma_{s-1})}$$

such losing branches, we have $M_k(\hat{\sigma}_s) \leq \frac{1}{2}$.

This completes the construction.

To verify that the construction works, note first that in every stage s, there is some prefix of σ_s where M_1 attains s. Hence $X \in S^{\infty}[M_1]$.

For every A-martingale M_i, $i \geq 1$, every prefix σ of X has some extension γ with $M_i(\gamma 0) \neq M_i(\gamma 1)$. By construction, there are infinitely many n with $M(X \upharpoonright n) \leq 1/2$. Thus $X \notin S^{\infty}_{\mathrm{str}}[M_i]$. □

6 Unique betting ratio against finitely many betting ratios

In this section, we consider finite rational ratio sets whose minimum is greater than 0 and whose maximum, less than 1. We introduce a notion which we use to compare two sets of rational numbers, whether finite or infinite.

Definition 11. Let A and B be possibly infinite sets of rationals in $(0, 1)$ such that their suprema are rationals less than 1 and infima are rationals greater than 0. We say that the ratio set A *majorizes* B if $\sup(A) > \sup(B)$.

Theorem 12. *Let A and B be finite sets of rationals in $(0, 1)$ such that their maxima are less than 1 and minima greater than 0. If A majorizes B, then $S^{\infty}[B] \subseteq S^{\infty}[A]$.*

Proof. Let $b_1 = \min(B)$ and $b_2 = \max(B)$, and let M be a B-martingale. Let a be an element of A such that $b_2 < a < 1$. We show that for a fixed set B, for every sequence X on which M succeeds, there is an A-martingale which succeeds on X.

First, assume that M bets only using b_1 and b_2.

Consider three A-martingales, N_1, N_2, and N_3 defined by

$$N_1(\sigma\beta) = \begin{cases} aN_1(\sigma) & \text{if } M(\sigma\beta) < M(\sigma) \\ (2-a)N_1(\sigma) & \text{otherwise,} \end{cases}$$

$$N_2(\sigma\beta) = \begin{cases} aN_2(\sigma) & \text{if } M(\sigma\beta) = b_1 M(\sigma) \text{ or } (2-b_2)M(\sigma) \\ (2-a)N_2(\sigma) & \text{otherwise,} \end{cases}$$

$$N_3(\sigma\beta) = \begin{cases} aN_3(\sigma) & \text{if } M(\sigma\beta) = b_2 M(\sigma) \text{ or } (2-b_1)M(\sigma) \\ (2-a)N_3(\sigma) & \text{otherwise.} \end{cases}$$

In other words, N_1 is an A-martingale that "imitates" M. The A-martingale N_2 agrees with M wherever the latter bets b_1 or $(2-b_1)$ of its capital, and disagrees elsewhere. Symmetrically, N_3 agrees with M exactly where the latter bets b_2 or $(2-b_2)$ of its capital.

Assume that $M(X \upharpoonright n) > r > 1$. We show that there is some constant c such that at least one of $N_1(X \upharpoonright n)$, $N_2(X \upharpoonright n)$, $N_3(X \upharpoonright n)$ and $N_4(X \upharpoonright n)$ exceeds r^c.

Suppose $M(X \upharpoonright n) = b_1^{j_1} b_2^{j_2} (2-b_1)^{i_1} (2-b_2)^{i_2} > r$. Note that either one of $b_1^{j_1}(2-b_1)^{i_1}$ or $b_2^{j_2}(2-b_2)^{i_2}$ has to be at least $\sqrt{r}$, in particular, greater than 1. There are the following three cases.

Case I. Suppose $b_1^{j_1}(2-b_1)^{i_1}$ and $b_2^{j_2}(2-b_2)^{i_2}$ are both greater than 1, and assume, without loss of generality, that $b_1^{j_1}(2-b_1)^{i_1} \geq \sqrt{r}$. Then we have

$$N_1(X \upharpoonright n) = a^{j_1+j_2}(2-a)^{i_1+i_2} > a^{j_1}(2-a)^{i_1}.$$

Since $b_1^{j_1}(2-b_1)^{i_1} > \sqrt{r}$, by Lemma 6, we have $a^{j_1}(2-a)^{i_1} > \sqrt{r}^c$, where $c = \left[\frac{\log(2-a)}{\log(2-b_1)}\right] > 0$.

Case II. Suppose $b_2^{j_2}(2-b_2)^{i_2} = \theta < 1$. Then it follows that $b_1^{j_1}(2-b_1)^{i_1} > \frac{r}{\theta} > r > 1$. Then we have

$$N_2(X \upharpoonright n) = a^{j_1+i_2}(2-a)^{i_1+j_2} \quad > \quad a^{j_1}(2-a)^{i_1}.$$

By the argument in case I, we can conclude that

$$a^{j_1}(2-a)^{i_1} > (r/\theta)^c,$$

where $c = [\log(2-a)/\log(2-b_1)]$. Hence, $N_2(X \upharpoonright n) > r^c$.

Case III. If $b_1^{j_1}(2-b_1)^{i_1} < 1$, then analogous to case II, we conclude that $N_3(X \upharpoonright n) > r^c$.

Since for each r, one of N_1, N_2 or N_3 attains capital r^c for some fixed constant $c > 0$, it follows by the pigeonhole principle that there is an i, $1 \leq i \leq 3$ such that

$$\limsup_{n\to\infty} N_i(X \upharpoonright n) = \infty.$$

Next, we show that the result holds if B has more than 2 values. Inductively, assume that if B contains n distinct values, and $\max(B) < \max(A)$, then there is a finite number of A-martingales which cover $S^\infty[B]$.

Let B' be a ratio set majorized by A, such that B' contains $n+1$ elements. Let M be a B'-martingale, and assume that

$$M(X \upharpoonright n) = \prod_{b \in B'} b^{j_b}(2-b)^{i_b+j_b} > r.$$

First, consider the case that there is a maximal proper subset $\mathcal{D}$ of B' such that

$$\prod_{b \in \mathcal{D}} b^{j_b}(2-b)^{i_b+j_b} > r,$$

and for each $b \in B' - \mathcal{D}$,

$$b^{j_b}(2-b)^{i_b+j_b} < 1.$$

By the inductive hypothesis, there is an A-martingale $\hat{N}_{\mathcal{D}}$ which bets on X, such that its capital restricted to those positions where M bets using ratios from $\mathcal{D}$, is at least r^c. Then the A-martingale $N_{\mathcal{D}}$ defined by

$$N_{\mathcal{D}}(\sigma 0) = \begin{cases} \hat{N}_{\mathcal{D}}(\sigma 0) & \text{if } \left\{\frac{M(\sigma 0)}{M(\sigma)}, \frac{M(\sigma 1)}{M(\sigma)}\right\} \in \{b, 2-b\} \text{ and } b \in \mathcal{D} \\ aN_{\mathcal{D}}(\sigma) & \text{if } M(\sigma 0) = (2-b)M(\sigma) \text{ and } b \in B' - \mathcal{D} \end{cases}$$

attains more than r^c on $X \upharpoonright n$, by the inductive hypothesis.

Otherwise, for every $b \in B'$, $b^{j_b}(2-b)^{i_b+j_b} \geq 1$. In this case, the A-martingale which bets $(2-a)$ of its capital when M bets more than 1 on a bit, and a elsewhere, makes at least r^c on $X \upharpoonright n$, where $c = \log(2-a)/\log(2-\max(B))$.

Thus for a fixed finite ratio set B majorized by a finite ratio set A, every sequence in $S^\infty[B]$ can be covered using finitely many A-martingales. Hence, we have $S^\infty[B] \subseteq S^\infty[A]$. $\square$

Note that the above theorem also implies that the inclusion is strict.

Corollary 13. *Let A and B be finite ratio sets such that A majorizes B. Then $S^\infty[B] \subsetneq S^\infty[A]$.*

Proof. Let $b = \max(B)$, and $a = \max(A)$. By assumption, $0 < b < a < 1$. Consider the valid ratio set $C = \{c\}$ where $b < c < a$. By Theorem 12, we know that $S^\infty[B] \subseteq S^\infty[C]$. Further, by Lemma 9, we have that $S^\infty[C] \subsetneq S^\infty[A]$, thus establishing the result. $\square$

7 Necessary Conditions for Dominance

Let A and B be finite sets of rationals. We conclude by showing that unless A majorizes B, $S^\infty[A]$ is not a superset of $S^\infty[B]$. We establish that if $\max A = \max B$, then $S^\infty[A] = S^\infty[B]$.

Theorem 14. *Let A and B be finite valid ratio sets which do not majorize each other. Then $S^\infty[A] = S^\infty[B]$.*

Proof. Let $b = \max(B) = \max(A)$. Define $B' = B - \{b\}$.

Let M be a B-martingale that succeeds on an infinite binary sequence X. Suppose that

$$M(X \upharpoonright n) > r > 1.$$

Partition the positions in $\{0, \ldots, n-1\}$ into two disjoint sets S_1 and S_2, where S_1 is the set of those positions where M bets using ratios from $\overline{B'}$ and S_2 is the set of those positions where M bets ratios from $\{b, 2-b\}$. Further, assume that

$$M(X \upharpoonright S_1) = r_1,$$

and

$$M(X \upharpoonright S_2) = r_2.$$

Then, by the proof of Theorem 12, there is an A-martingale N such that

$$N(X \upharpoonright S_1) > r_1^c.$$

where $0 < c < 1$. Define the martingale $\hat{N}$ by

$$\hat{N}(\lambda) = 1$$

$$\hat{N}(wb) = \begin{cases} \frac{N(wb)}{N(w)}\hat{N}(w) & \text{if } \frac{N(wb)}{N(w)} \in \overline{B'} \\ b\hat{N}(w) & \text{if } \frac{N(wb)}{N(w)} = b \\ (2-b)\hat{N}(w) & \text{if } \frac{N(wb)}{N(w)} = 2-b \end{cases}$$

It is routine to see that $\hat{N}$ is a computable A-martingale which succeeds in earning at least $r_1^c r_2 > r^c$ on X, where $c > 0$.

If $X \in S^\infty[M]$, then it follows that $X \in S^\infty[\hat{N}]$. Hence $S^\infty[B] \subseteq S^\infty[A]$. By a symmetric argument, we see that $S^\infty[A] \subseteq S^\infty[B]$. $\square$

8 Open Questions

The most important open question is, for an infinite set of ratios B and a finite or an infinite set of ratios A, if A majorizes B, does it follow that $S^\infty[B] \subsetneq S^\infty[A]$? The proofs in our work rely on the fact that A and B are finite.

The construction in Theorem 9 shows that if A majorizes B, then there is a sequence $X \in S^\infty[A] - S^\infty[B]$. It is interesting to consider the minimal complexity of such a sequence.

Another question which is open is whether restricted wagers, or restricted ratios can be used to define refinements of the notion of effective Hausdorff and effective packing dimension of sequences. It is possible that it leads to a new quantification of the information in infinite sequences.

Acknowledgments

The authors wish to thank Laurent Bienvenu, Rod Downey, Wolfgang Merkle and Ron Peretz for helpful discussions.

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Appendix

Lemma 15. *The function $f : (0,1) \to \mathbb{R}$ defined by $f(x) = \log_{2-x}(x)$ is monotone increasing.*

Proof. Let $b < a$. We need to show that

$$\log_{2-b}(b) < \log_{2-a}(a).$$

Consider the function $g : (0,1) \to \mathbb{R}$ defined by $g(x) = \log_{2-x}(x)$. Then

$$\frac{dg(x)}{dx} = \frac{d}{dx}\left(\frac{\ln x}{\ln(2-x)}\right) = \frac{\frac{\ln(2-x)}{x} + \frac{\ln x}{(2-x)}}{(\ln(2-x))^2}.$$

If we show that this quantity is non-negative, then g is monotone increasing, and hence f is monotone increasing.

The above quantity is non-negative if and only if the function $h : (0,1) \to \mathbb{R}$ defined by

$$h(x) = (2-x)\ln(2-x) + x\ln x$$

is non-negative.

Now,

$$\frac{dh(x)}{dx} = -1 - \ln(2-x) + 1 + \ln x = \ln x - \ln(2-x) < 0.$$

Hence h is monotone decreasing in $(0,1)$.

We have that

$$\lim_{x\to 0} x\ln x = \lim_{x\to 0} \frac{\ln x}{1/x} = \lim_{x\to 0} \frac{1/x}{-1/x^2} = 0.$$

Thus $h(1^-) = 1\ln 1 + 0 = 0$. We can conclude that h is positive in $(0,1)$, hence g is monotone increasing in $(0,1)$ and thus f is monotone increasing in $(0,1)$. □