A CHARACTERIZATION OF CONSTRUCTIVE DIMENSION

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Outline

1 Introduction to Algorithmic Randomness
   - Constructive Measure Theoretic Approach
   - Martingales
   - Measures of Impossibility
   - Questions
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8. Characterization of Constructive Dimension
We consider a finite alphabet $\Sigma = \{0, 1\}$. The space of infinite binary sequences drawn from the alphabet is denoted $\mathcal{C}$. 

Martin-Löf in 1966 defined the notion of an effective measure-$0$ set. 

**Theorem (Martin-Löf 1966)** For every computable probability measure $P$ defined on $\mathcal{C}$, there is a unique largest effective measure-$0$ set. The complement of the largest effective measure-$0$ set is the set of individual constructively random sequences.
We consider a finite alphabet $\Sigma = \{0, 1\}$. The space of infinite binary sequences drawn from the alphabet is denoted $\mathcal{C}$.

**Definition**

A *probability measure* on $\mathcal{C}$ is a function $P : \Sigma^* \rightarrow [0, 1]$ satisfying

1. $P(\lambda) = 1$,
2. For every $w \in \Sigma^*$, $P(w) = P(w0) + P(w1)$. 

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* (Martin-Löf 1966) For every computable probability measure $P$ defined on $C$, there is a unique largest effective measure-0 set.

The complement of the largest effective measure-0 set is the set of **individual constructively random sequences**.
Martingale Approach to Randomness

**Definition**

(Ville 1939, Schnorr 1970,1971) A *martingale* is a function \( d : \Sigma^* \rightarrow [0, \infty) \) such that the following hold.

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d(\lambda) \leq 1.
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**Definition**

The martingale **succeeds** on a set $X \subseteq \mathcal{C}$ if

$$(\forall \omega \in X) \limsup_{n \to \infty} d(\omega[0 \ldots n-1]) = \infty.$$

The success set of a martingale is denoted $S^\infty[d]$. 

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**Legends:**

- **$\lambda$** represents the empty string.
- **$\Sigma^*$** denotes the set of all finite strings over the alphabet $\Sigma$.
- **$\mathcal{C}$** represents the set of all infinite strings over the alphabet $\Sigma$.
- **$P$** denotes the probability measure.
- **$d$** is the martingale function.
- **$\omega$** represents a string in $\mathcal{C}$.
- **$\omega[0 \ldots n-1]$** denotes the prefix of the string $\omega$ up to the $(n-1)$-th character.
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The success set of a martingale is denoted \( S^\infty[d] \).

A sequence \( \omega \in C \) is random if and only if there is a constructive martingale that succeeds on it.
Definition

(Gács 81, Vovk, V’yugin 93) A measure of impossibility with respect to a computable probability measure $P$ is a function $p : C \rightarrow [0, \infty]$ such that the following hold.

- $p$ is lower semicomputable.
- $\int p \, dP \leq 1$.

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Measure of Impossibility Approach to Randomness

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QUESTION:
Can measures of impossibility be generalized to characterize resource-bounded dimension?
**Definition**

Let $\Omega$ be $\mathbb{C}$ or $\Sigma^*$. A function $f : \Omega \rightarrow [-\infty, \infty]$ is called *lower semicomputable* if $S = \{(w, q) | w \sqsubseteq x, q < f(x)\}$ is the union of a computably enumerable sequence of cylinders in the natural topology on $\Omega \times \mathbb{Q}$.

A function $f : \Omega \rightarrow [-\infty, \infty]$ is *upper semicomputable* if $-f$ is lower semicomputable. A function $f$ is *computable* if it is both upper and lower semicomputable.

A probability measure on $\mathbb{C}$ is computable if $P : \Sigma^* \rightarrow [0, 1]$ is a computable function.
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Converting a martingale into a measure of impossibility

Let \( d : \Sigma^* \to [0, \infty) \) be a lower semicomputable \( P \)-martingale that succeeds on an \( \omega \in C \).

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**Definition**

A martingale \( d \) succeeds strongly on \( X \) if \((\forall \omega \in X) \lim \inf_{n \to \infty} d(\omega[0..n-1]) = \infty\).

**The strong success set of a martingale** \( d \) **is denoted** \( S_{\infty}^{\text{str}}[d] \).

**Lemma** (folklore) Let \( \omega \in S_{\infty}^{\text{str}}[d] \). Then there exists a martingale \( d' : \Sigma^* \to [0, \infty] \) such that \( \omega \in S_{\infty}^{\text{str}}[d'] \).

**Proof** is by the "savings account trick".

Define \( p : C \to [0, \infty] \) by \( p(\omega) = \lim_{n \to \infty} sa(\omega[0..n-1]) \).
Let $d : \Sigma^* \to [0, \infty]$ be a lower semicomputable $P$-martingale that succeeds on an $\omega \in C$.

**Definition**

A martingale $d$ succeeds strongly on $X$ if

$$\left( \forall \omega \in X \right) \lim_{n \to \infty} \inf d(\omega[0 \ldots n - 1]) = \infty.$$  

The strong success set of a martingale $d$ is denoted $S^{\infty}_{str}[d]$.

**Lemma**

*(folklore)* Let $\omega \in S^{\infty}[d]$. Then there exists a martingale $d' : \Sigma^* \to [0, \infty]$ such that $\omega \in S^{\infty}_{str}[d']$.

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Let \( d : \Sigma^* \rightarrow [0, \infty] \) be a lower semicomputable \( P \)-martingale that succeeds on an \( \omega \in C \).

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**Lemma**

*(folklore)* Let \( \omega \in S^\infty[d] \). Then there exists a martingale \( d' : \Sigma^* \rightarrow [0, \infty] \) such that \( \omega \in S_{\text{str}}^\infty[d'] \).

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Define \( p : C \rightarrow [0, \infty] \) by

\[
p(\omega) = \lim_{n \to \infty} sa(\omega[0 \ldots n - 1]).
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Let \( C_w = \{ \omega \mid \omega \in C, w \sqsubseteq \omega \} \), and \( \omega_n = \omega[0\ldots n - 1] \).

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\int pdP = \int \lim_{n \to \infty} sa(\omega[0 \ldots n - 1])
\leq \liminf_{n \to \infty} \int_{C_{\omega_n}} sa(\omega_n) dP \quad \text{[Fatou’s Lemma]}
\]

\[
= \liminf_{n \to \infty} sa(\omega_n) P(\omega_n)
\leq \liminf_{n \to \infty} d(\omega_n) P(\omega_n)
\leq 1. \quad \text{[Kraft’s Inequality]}
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- $p(\omega) = \infty$ if $d'$ strongly succeeds on $\omega$. 
We consider *strongly positive probability measures*: There exists a computable function $h : \Sigma^* \times 0^\mathbb{N} \rightarrow \mathbb{Q}^+$ such that if $P(w) \neq 0$ and $\hat{P} : \Sigma^* \times 0^\mathbb{N} \rightarrow \mathbb{Q}$ is a witness to the computability of $P$, then for all $n \in \mathbb{N}$, $h(w, 0^n) < P(w)$. 

Let $p : \Sigma^\infty \rightarrow [0, \infty]$ be a $P$-measure of impossibility. Then define $d : \Sigma^* \rightarrow [0, \infty)$ by

$$d(wb) =
\begin{cases}
\int_{\omega} \frac{p(\omega)}{dP} dP(w) & \text{if } P(w) > 0,
2 & \text{otherwise}.
\end{cases}$$
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Let $p : \Sigma^\infty \to [0, \infty]$ be a $P$-measure of impossibility. Then define $d : \Sigma^* \to [0, \infty)$ by

$$d(wb) = \begin{cases} \frac{\int_{cwb} p(\omega)dP}{P(wb)} & \text{if } P(wb) > 0, \\ 2d(w) & \text{otherwise.} \end{cases}$$
Converting a Measure of Impossibility to a Martingale

- $d$ is a martingale by linearity of expectation, and $\int pdP \leq 1$. 


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Thus $d$ is a lower semicomputable martingale that succeeds on $\omega$. 

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- $d$ is lower semicomputable with a computable monotone sequence of integrals of step functions converging to the value of $d$.
- $p(\omega) = \infty$ implies $\limsup_{n \to \infty} d(\omega_n) = \infty$. The proof uses the lower semicomputability of $p$. 
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- $d(\lambda) \leq 1$.
- For all $w \in \Sigma^*$, $d(w)P^s(w) = d(w0)P^s(w0) + d(w1)P^s(w1)$. 

Definition

An $s$-gale $d$ is said to succeed on a set $X \subseteq C$ if

\[ \forall \omega \in X \limsup_{n \to \infty} d(\omega_n) = \infty. \]

An $s$-gale $d$ is said to succeed strongly on a set $X \subseteq C$ if

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Constructive Dimensions

Definition

Let

\[ G^{\text{constr}}(X) = \{ s : \text{there is a constructive } s\text{-gale } d, X \subseteq S^{\infty}[d] \}. \]

The constructive Hausdorff dimension (Lutz 2000) is

\[ \dim^{\text{constr}}(X) = \inf G^{\text{constr}}(X). \]
Constructive Dimensions

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The constructive Hausdorff dimension (Lutz 2000) is

$$dim^{constr}(X) = \inf G^{constr}(X).$$

Let

$$G^{constr}_{str}(X) = \{s : \text{ there is a constructive } s\text{-gale } d, X \subseteq S^\infty_{str}[d]\}.$$ 

The constructive packing dimension (Athreya et al. 2004) is

$$Dim^{constr}(X) = \inf G^{constr}(X).$$
s-Improbability

Definition
A set $X$ is said to be $s$-improbable with respect to a strongly positive probability measure $P$ if there is a measure of impossibility $p : \Omega \rightarrow [0, \infty]$ such that

$$\forall \omega \in X, \limsup_{n \rightarrow \infty} \frac{\int_{C_{\omega_n}} pdP}{Ps(\omega_n)} = \infty.$$
s-Improbability

**Definition**

A set $X$ is said to be *s-improbable with respect to a strongly positive probability measure $P$* if there is a measure of impossibility $p : \Omega \rightarrow [0, \infty]$ such that

$$\forall \omega \in X, \limsup_{n \rightarrow \infty} \frac{\int_{C_{\omega_n}} pdP}{P^s(\omega_n)} = \infty.$$ 

**Definition**

A set $X \subseteq C$ is said to be *strongly s-improbable with respect to a strongly positive probability measure $P$* if there is a measure of impossibility $p : \Omega \rightarrow [0, \infty]$ such that

$$\forall \omega \in X, \liminf_{n \rightarrow \infty} \frac{\int_{C_{\omega_n}} pdP}{P^s(\omega_n)} = \infty.$$
Summary of Constructions

Martingale to measure of impossibility

\[ p(\omega) = \lim_{n \to \infty} sa(\omega_n). \]

s-gale to s-measure of improbability

\[ p(\omega) = \lim_{n \to \infty} sa(\omega_n)P^{s-1}(\omega_n). \]
Summary of Constructions

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\[ p(\omega) = \lim_{n \to \infty} sa(\omega_n). \]

Measure of Impossibility to Martingales

\[ d(w) = \frac{\int_{C_w} pdP}{P(w)} \]

s-gale to s-measure of improbability

\[ p(\omega) = \lim_{n \to \infty} sa(\omega_n) P^{s-1}(\omega_n). \]

s-measure of Impossibility to s-gales

\[ d(w) = \frac{\int_{C_w} pdP}{P^s(w)}. \]
Lemma

Let $s \in [0, \infty)$. $X$ is $s$-improbable with respect to $P$ if and only if $s \in G_{\text{constr}}(X)$, and $X$ is strongly $s$-improbable with respect to $P$ if and only if $s \in G_{\text{str}}(X)$. 
Alternate Characterization of Constructive Dimension

Theorem

\[ \dim^{\text{constr}}(X) = \inf\{s \mid X \text{ is } s\text{-improbable with respect to } P\} \]

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Alternate Characterization of Constructive Dimension

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*Thank You.*