A CHARACTERIZATION OF CONSTRUCTIVE DIMENSION

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Outline

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Constructive Measure Theoretic Approach to Randomness

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A *probability measure* on **C** is a function $P : \Sigma^* \longrightarrow [0, 1]$ satisfying

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Theorem

(Martin-Löf 1966) For every computable probability measure P defined on $\mathbf{C}$, there is a unique largest effective measure-0 set.

The complement of the largest effective measure-0 set is the set of *individual constructively random sequences.*

Martingale Approach to Randomness

Definition

(Ville 1939, Schnorr 1970,1971) A *martingale* is a function $d : \Sigma^* \to [0, \infty)$ such that the following hold.

$$d(\lambda) \leq 1.$$

$$(\forall w \in \Sigma^*)\ d(w)P(w) = d(w0)P(w0) + d(w1)P(w1).$$

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The martingale *succeeds* on a set $X \subseteq \mathbf{C}$ if

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A sequence $\omega \in \mathbf{C}$ is random if and only if there is a constructive martingale that succeeds on it.

Measure of Impossibility Approach to Randomness

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(Gács 81, Vovk, V'yugin 93) A *measure of impossibility* with respect to a computable probability measure P is a function $p : \mathbf{C} \rightarrow [0, \infty]$ such that the following hold.

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A sequence $\omega \in \mathbf{C}$ is random if and only if there is a measure of impossibility p such that $p(\omega) = \infty$.

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QUESTION:
Can measures of impossibility be generalized to characterize resource-bounded dimension?

Definition

Let Ω be $\mathbf{C}$ or Σ^*. A function $f : \Omega \to [-\infty, \infty]$ is called *lower semicomputable* if $S = \{(w, q) | w \sqsubseteq x, q < f(x)\}$ is the union of a computably enumerable sequence of cylinders in the natural topology on $\Omega \times \mathbb{Q}$.

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A function $f : \Omega \to [-\infty, \infty]$ is *upper semicomputable* if $-f$ is lower semicomputable. A function f is *computable* if it is both upper and lower semicomputable.

A probability measure on $\mathbf{C}$ is computable if $P : \Sigma^* \to [0, 1]$ is a computable function.

Converting a martingale into a measure of impossibility

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A martingale d *succeeds strongly* on X if

$$(\forall \omega \in X) \ \liminf_{n \to \infty} d(\omega[0 \ldots n-1]) = \infty.$$

The strong success set of a martingale d is denoted $S^{\infty}_{str}[d]$.

Lemma

(folklore) Let $\omega \in S^{\infty}[d]$. Then there exists a martingale $d' : \Sigma^* \to [0, \infty]$ such that $\omega \in S^{\infty}_{str}[d']$.

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Define $p : \mathbf{C} \to [0, \infty]$ by

$$p(\omega) = \lim_{n \to \infty} sa(\omega[0 \ldots n-1]).$$

Converting a Martingale into a Measure of Impossibility

Let $\mathbf{C}_w = \{\omega \mid \omega \in \mathbf{C}, w \sqsubseteq \omega\}$, and $\omega_n = \omega[0 \ldots n-1]$.

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\begin{aligned}
\int p dP &= \int \lim_{n\to\infty} sa(\omega[0\ldots n-1]) \\
&\leq liminf_{n\to\infty} \int_{\mathbf{C}_{\omega_n}} sa(\omega_n) dP && \text{[Fatou's Lemma]} \\
&= liminf_{n\to\infty} sa(\omega_n) P(\omega_n) \\
&\leq liminf_{n\to\infty} d(\omega_n) P(\omega_n) \\
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&\leq 1. \qquad \text{[Kraft's Inequality]}
\end{aligned}$$
- $p(\omega) = \infty$ if d' strongly succeeds on ω.

Converting a measure of impossibility to a martingale

We consider *strongly positive probability measures*: There exists a computable function $h : \Sigma^* \times 0^{\mathbb{N}} \to \mathbb{Q}^+$ such that if $P(w) \neq 0$ and $\hat{P} : \Sigma^* \times 0^{\mathbb{N}} \to \mathbb{Q}$ is a witness to the computability of P, then for all $n \in \mathbb{N}$, $h(w, 0^n) < P(w)$.

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Let $p : \Sigma^\infty \to [0, \infty]$ be a P-measure of impossibility. Then define $d : \Sigma^* \to [0, \infty)$ by

$$d(wb) = \begin{cases} \frac{\int_{\mathsf{C}_{wb}} p(\omega) dP}{P(wb)} & \text{if } P(wb) > 0, \\ 2d(w) & \text{otherwise.} \end{cases}$$

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- $p(\omega) = \infty$ implies $\limsup_{n \to \infty} d(\omega_n) = \infty$. The proof uses the lower semicomputability of p.

Thus d is a lower semicomputable martingale that succeeds on ω.

s-gales

Definition

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Definition

An s-gale d is said to *succeed* on a set $X \subseteq \mathbf{C}$ if

$$(\forall \omega \in X) \limsup_{n \to \infty} d(\omega_n) = \infty.$$

An s-gale d is said to *succeed strongly* on a set X if

$$(\forall \omega \in X) \liminf_{n \to \infty} d(\omega_n) = \infty.$$

Constructive Dimensions

Definition

Let

$$G^{constr}(X) = \{s : \text{ there is a constructive } s\text{-gale } d, X \subseteq S^{\infty}[d]\}.$$

The constructive Hausdorff dimension (Lutz 2000) is

$$dim^{constr}(X) = \inf G^{constr}(X).$$

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Let

$$G^{constr}_{str}(X) = \{s : \text{ there is a constructive } s\text{-gale } d, X \subseteq S^{\infty}_{str}[d]\}.$$

The constructive packing dimension (Athreya et al. 2004) is

$$Dim^{constr}(X) = \inf G^{constr}(X).$$

s-Improbability

Definition

A set X is said to be *s-improbable with respect to a strongly positive probability measure P* if there is a measure of impossibility $p : \Omega \to [0, \infty]$ such that

$$\forall \omega \in X, \limsup_{n \to \infty} \frac{\int_{\mathbf{C}_{\omega_n}} p dP}{P^s(\omega_n)} = \infty.$$

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Definition

A set $X \subseteq C$ is said to be *strongly s-improbable with respect to a strongly positive probability measure P* if there is a measure of impossibility $p : \Omega \to [0, \infty]$ such that

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Summary of Constructions

Martingale to measure of impossibility

$$p(\omega) = \lim_{n\to\infty} sa(\omega_n).$$

s-gale to s-measure of improbability

$$p(\omega) = \lim_{n\to\infty} sa(\omega_n)P^{s-1}(\omega_n).$$

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Martingale to measure of impossibility

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Measure of Impossibility to Martingales

$$d(w) = \frac{\int_{\mathbf{C}_w} p dP}{P(w)}$$

s-gale to *s*-measure of improbability

$$p(\omega) = \lim_{n \to \infty} sa(\omega_n) P^{s-1}(\omega_n).$$

s-measure of Impossibility to *s*-gales

$$d(w) = \frac{\int_{\mathbf{C}_w} p dP}{P^s(w)}.$$

Lemma

Let $s \in [0, \infty)$. X is s-improbable with respect to P if and only if $s \in G^{constr}(X)$, and X is strongly s-improbable with respect to P if and only if $s \in G^{constr}_{str}(X)$.

Alternate Characterization of Constructive Dimension

Theorem

$$\begin{aligned} dim^{constr}(X) &= \inf\{s \mid X \textit{ is s-improbable with respect to } P\}. \\ Dim^{constr}(X) &= \inf\{s \mid X \textit{ is strongly s-improbable with} \\ &\qquad \textit{respect to } P\}. \end{aligned}$$

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Thank You.