A Characterization of Constructive Dimension

Satyadev Nandakumar
Department of Computer Science
Iowa State University
Ames, Iowa 50011
U.S.A.

December 10, 2008

Abstract

In the context of Kolmogorov's algorithmic approach to the foundations of probability, Martin-Löf defined the concept of an individual random sequence using the concept of a constructive measure 1 set. Alternate characterizations use constructive martingales and measures of impossibility. We prove a direct conversion of a constructive martingale into a measure of impossibility and vice versa, such that their success sets, for a suitably defined class of computable probability measures, are equal. The direct conversion is then generalized to give a new characterization of constructive dimensions, in particular, the constructive Hausdorff dimension and the constructive packing dimension, and their generalizations, the constructive scaled dimension and the constructive scaled strong dimension.

1 Introduction

One of the prime successes of the algorithmic approach to the foundations of probability theory envisioned by Kolmogorov is Martin-Löf's definition of an individual random sequence [15] using constructive measure theory. The measure-theoretic approach to the definition of random sequences identifies a property of "typical" sets. A random sequence is one that belongs to every reasonable majority of sequences [7]. The notion of a reasonable majority is formulated as an effective version of measure 1. Each measure 1 set has a complement set of measure 0. It is hence sufficient to define the concept of the effective measure zero set.

Let P be a computable probability measure defined on the Cantor Space (defined in section 3). For finite strings x, we consider cylinders C_x, the set of all infinite sequences with x as a prefix. A set S of sequences from the sample space of all sequences has P-measure zero if, for each $\varepsilon > 0$, there is a sequence of cylinder sets $C_{x_0}, C_{x_1}, \ldots, C_{x_i}, \ldots$ such that

$$S \subseteq \cup_i C_{x_i} \text{ and } P(\cup_i C_{x_i}) < \varepsilon.$$

A set of sequences S has effective P-measure zero if there is a computable function $h(i, \varepsilon)$ such that $h(i, \varepsilon) = C_{x_i}$ for each i.

Martin-Löf proved a universality property – that for any computable measure P, there is a unique largest effective P-measure zero set. The elements in the complement of this set are the set of P-random sequences.

Another tool in the study of effective randomness is the concept of martingales. Introduced by Ville in the 1930s [21] (being implicit in the early work of Lévy [8], [9]), they were applied in theoretical computer science by Schnorr in the early 1970s [16], [17], [18] in his investigation of Martin-Löf randomness, and by Lutz [11], [12], [13] in the development of resource-bounded measure. A martingale is a betting strategy, which, for a probability measure P defined on the Cantor Space, obeys the conditions,

$$\begin{aligned} d(\lambda) &\leq 1 \\ d(w)P(w) &= d(w0)P(w0) + d(w1)P(w1). \end{aligned} \tag{1}$$

Intuitively, it can be seen as betting strategy on an infinite sequence, where, for each prefix w of the infinite sequence, the amount $d(w)$ is the capital that is in hand after betting. A martingale can be seen as a fair betting condition where the expected value after every bet is the same as the expected value before the bet is made. It is said to succeed on a sequence ω if

$$\limsup_{n \to \infty} d(\omega[0 \ldots n-1]) = \infty.$$

The success set of a martingale, $S^\infty[d]$, is the set of all individual sequences on which it succeeds. In this work, we consider constructive martingales. There is a universal martingale $\tilde{d}$ which is constructive and for every ω, if there is a constructive martingale d which succeeds on ω, then $\tilde{d}$ succeeds on ω. The theory of martingales and their applications to the field of resource-bounded measure, complexity theory, and resource-bounded dimension has proved to be remarkably fruitful. In this work, we wish to establish connections between martingales and a third approach of defining randomness, *viz.*, that of a measure of impossibility.

This third approach to define a random sequence is to characterize a degree of disagreement between any sequence ω and the probability P. Following [22], a *measure of impossibility* is a positive function $p(\omega)$ which describes the quantitative level to which ω is impossible with respect to the probability measure P. A measure of impossibility is defined to be a lower semicomputable function p, which is integrable with respect to P. It can be seen that if ω is P-random, then $p(\omega) < \infty$. There is an optimal measure of impossibility $\tilde{p}$ such that a sequence ω is random if and only if $\tilde{p}(\omega) < \infty$ [22], [23]. This concept is a central one in V'yugin's proof of an effective version of the Ergodic Theorem [23].

The relation between martingales and Martin-Löf's definition of randomness was studied by Schnorr [16], [17], [18]. The proof that the notion of randomness defined by Martin-Löf corresponds to that of the ones defined via the measures of impossibility is due to Vovk and V'yugin [22] and V'yugin [23]. We establish a direct correspondence between the notions of constructive martingales and measures of impossibility.

We then apply this construction to come up with an analogous new definition of constructive dimension [14] in terms of a generalized version of the notion of a measure of impossibility. We show that this construction also generalizes to give an alternate definition of constructive scaled dimension [4].

The main difference between a proof based on martingales and one using a measure of impossibility is that a martingale is defined on the basis cylinders, and a measure of impossibility is a pointwise notion. Measure of impossibility seems to be an easier tool in dealing with theorems in which we have to reason about the convergence of general random variables defined on the points in the sample space. However, we show that at the constructive level, these tools are equivalent. Since there are universal objects available in both the cases, there exists an indirect conversion between the two such that the success set of a martingale can be converted into that of a measure of impossibility and conversely; this work contributes a direct constructive conversion of one into another. The theory of algorithmic randomness has been remarkably fruitful to date. (For a survey of the field, see [2].) Martingales have proved to have greater apparent utility in some cases than Martin-Löf tests in studying randomness, and measures of impossibility have been of use in establishing a remarkable result in the study of algorithmic randomness, the Effective Ergodic Theorem [23] . We hope that the explicit transformation of this work will improve the understanding, and perhaps the utility of measure of impossibility. Moreover, in the absence of universal objects which happens at computable and other levels a conversion of this nature may be useful.

2 Preliminaries

We consider the binary alphabet $\Sigma = \{0, 1\}$. The empty string is denoted by λ. The set of finite strings from the alphabet is denoted as Σ^* and the set of infinite sequences, as $\mathbf{C}$, the Cantor Space. For finite strings y, x, and infinite sequences ω, we denote x to be a prefix of y or of ω as $x \sqsubseteq y$, or as $x \sqsubseteq \omega$, respectively. We adopt the convention, for all sequences and strings x, for all $0 \le n \le |x|$, the n-length prefix of x is denoted $x[0 \dots n-1]$ (this is always finite), and for $n > |x|$, we have $x[0 \dots n-1] = x$ by notation.

Following the common notation, $\mathbb{N}$ represents the set of natural numbers, $\mathbb{Q}$ the set of rational numbers and $\mathbb{Z}$ the set of integers. Denote $[-\infty, \infty]$ for $\mathbb{R} \cup \{-\infty, \infty\}$, $\mathbb{R}^+$ for the non-negative reals, and $[0, \infty]$ for $\mathbb{R}^+ \cup \{\infty\}$.

We define the notion of lower semicomputability for the natural topology on the product space $\mathbf{C} \times \mathbb{Q}$ or $\Sigma^* \times \mathbb{Q}$. The natural topology on $\mathbb{Q}$ or Σ^* is discrete (i.e., the topology made of the set of all subsets of $\mathbb{Q}$ or of Σ^*). The natural topology on $\mathbf{C}$ is generated by the cylinders $C_x = \{\omega \mid x \sqsubseteq \omega\}$, where $x \in \Sigma^*$. A function $f : \Sigma^* \cup \mathbf{C} \to [-\infty, \infty]$ is called lower semicomputable if its *graph* $S = \{(\omega, q) \mid \omega \in \Sigma^* \cup \mathbf{C}$ and $q \in \mathbb{Q}, q < f(\omega)\}$ is a union of a computably enumerable sequence of intervals in the natural topology on $\mathbb{Q} \times \Sigma^*$. The function f is lower semicomputable, if for any rational number q and any finite string w, the assertion $q < f(\omega)$ is true can be verified in a computable manner.

We prove an equivalent notion of lower semicomputability:

Lemma 1. *The following hold.*

i. A function $f : \mathbf{C} \to [-\infty, \infty]$ is lower semicomputable if and only if there exists a computable function $\hat{f} : \Sigma^* \times 0^{\mathbb{N}} \to \mathbb{Q} \cup \{-\infty, \infty\}$ such that the following hold: For all $\omega \in \mathbf{C}$,

(a) Monotonicity: For all $m, n \in \mathbb{N}$, $\hat{f}(\omega[0 \dots m-1], 0^n) \le \hat{f}(\omega[0 \dots m-1], 0^{n+1}) \le f(\omega)$, and $\hat{f}(\omega[0 \dots m-1], 0^n) \le \hat{f}(\omega[0 \dots m], 0^n) \le f(\omega)$.

(b) Convergence: We have

$$\lim_{n \to \infty} \hat{f}(\omega[0 \dots n-1], 0^n) = f(\omega).$$

ii. $f : \Sigma^* \to [-\infty, \infty]$ is lower semicomputable if and only if there exists a function $\hat{f} : \Sigma^* \times 0^{\mathbb{N}} \to \mathbb{Q} \cup \{-\infty, \infty\}$ such that the following hold: For all $x \in \Sigma^*$, *(a') Monotonicity: For all* $n \in \mathbb{N}$, $\hat{f}(x, 0^n) \leq \hat{f}(x, 0^{n+1}) \leq f(x)$.

(b') Convergence: $\lim_{n\to\infty} \hat{f}(x, 0^n) = f(x)$.

Proof. The characterization for the case when f is defined on the domain Σ^* is standard in the literature (see [14]), and we prove the formulation for the case when the domain is $\mathbf{C}$.

For the case when $f : \mathbf{C} \to [-\infty, \infty]$ is lower semicomputable, first assume that the set $S = \{(\omega, q) \mid \omega \in \mathbf{C}, q \in \mathbb{Q}, f(\omega) > q\}$ is the union of a computable enumeration $S : 0^{\mathbb{N}} \to \Sigma^* \times (\mathbb{Q} \cup \{-\infty\})$ of cylinders in the natural topology on $\mathbf{C} \times \mathbb{Q}$. The projection functions $\pi_1 : \Sigma^* \times \mathbb{Q} \to \Sigma^*$ and $\pi_2 : \Sigma^* \times \mathbb{Q} \to \mathbb{Q}$ are defined as $\pi_1(w, q) = w$ and $\pi_2(w, q) = q$. We design a witness function $\hat{f} : \Sigma^* \times 0^{\mathbb{N}} \to \mathbb{Q} \cup \{-\infty\}$ in the following algorithm.

procedure $\hat{f}(w,\, 0^n)$
  Set $\leftarrow \{-\infty\}$.
  i $\leftarrow 0$.
  while i $\leq n$ **do**
    if $\pi_1(S(i)) \sqsubseteq w$ **then**
      Set $\leftarrow$ Set $\cup \{\pi_2(S(i))\}$
    end if
    i $\leftarrow i + 1$.
  end while
  return max(Set)
end procedure

The monotonicity condition is satisfied, because in the algorithm, for every n, the sets have the following relationships:

$$\{\pi_2(S(i)) \mid \pi_1(S(i)) \sqsubseteq w, 0 \leq i \leq n\} \subseteq \{\pi_2(S(i)) \mid \pi_1(S(i)) \sqsubseteq w, 0 \leq i \leq n{+}1\},$$

and, for strings w' and w, if $w' \sqsubseteq w$, then

$$\{\pi_2(S(i)) \mid \pi_1(S(i)) \sqsubseteq w', 0 \leq i \leq n\} \subseteq \{\pi_2(S(i)) \mid \pi_1(S(i)) \sqsubseteq w, 0 \leq i \leq n\}.$$

For the convergence, it is obvious that

$$\lim_{n\to\infty} f(\omega[0 \ldots n-1], 0^n) \tag{2}$$

exists, since it is a monotone bounded sequence in a compact space. To see that the limit is $f(\omega)$, assume that the limit (2) is a real $r < f(\omega)$. Then there exists a rational r', $r < r' < f(\omega)$, such that there is no prefix w of ω such that (w, r') occurs in the enumeration of S. This is a contradiction. Hence the condition is satisfied, and limit (2) is $f(\omega)$.

Conversely, let $f : \mathbf{C} \to \overline{\mathbb{R}}$ be a lower semicomputable function with witness $\hat{f} : \Sigma^* \times 0^{\mathbb{N}} \to \mathbb{Q}$ satisfying lower semicomputability conditions. We prove that the set $S = \{(\omega, r) : r \in \mathbb{Q}, r < f(\omega)\}$ is the union of a computable enumeration of cylinders in $\mathbf{C} \times \mathbb{Q}$.

We show that for every $r \in \mathbb{Q}$, $r < f(\omega)$, there is a prefix w of ω, such that (w, r) is accepted by an algorithm. This is routine to see, since, we can dovetail the execution of $\hat{f}$ on $\Sigma^* \times 0^{\mathbb{N}}$. If $r < f(\omega)$, there is an $r' > r$ such that $\hat{f}$ produces r' on some prefix w and some 0^m, and then we can accept (w, r). □

A function f is called upper semicomputable if $-f$ is lower semicomputable. A function f is called computable if it is both lower and upper semicomputable. This may be seen to be equivalent to the following definition in the case of functions defined over Σ^*.

Definition 2. A function $f : \Sigma^* \to \mathbb{R}$ is said to be computable if there exists a function $\hat{f} : \Sigma^* \times 0^{\mathbb{N}} \to \mathbb{Q}$ such that for every $n \in \mathbb{N}$ and every $x \in \Sigma^*$,

$$|\hat{f}(x, 0^n) - f(x)| \leq 2^{-n}.$$

Note. It is easy to show that if $\hat{f}$ is a witness function to the computability of f, then for all n, $\hat{f}(x, 0^n) - 2 \cdot 2^{-n}$ is a lower semicomputation, and $\hat{f}(x, 0^n) + 2 \cdot 2^{-n}$ is an upper semicomputation of f.

3 Effective Randomness

Let $(\Omega, \mathcal{F}, P)$ be the probability space, where Ω is the sample space, $\mathcal{F}$ is the Borel σ-algebra (members of $\mathcal{F}$ are the *events*), and $P : \mathcal{F} \to [0, 1]$ is the probability. We will be concerned with the sample space $\Omega = \mathbf{C}$, the *Cantor Space*, the set of all infinite binary sequences. $\mathcal{F}$ is the σ-algebra generated by the cylinders $C_x = \{\omega \mid \omega \in \mathbf{C}, x \sqsubseteq \omega\}$.

Definition 3. A probability measure P defined on the Cantor Space of infinite binary sequences is characterized by the following:

1. $P(\lambda) = 1$.
2. For every string w, $0 \leq P(w) = P(w0) + P(w1)$.

A probability measure $P : \mathcal{F} \to [0, 1]$ is called computable if the probability measure $P : \Sigma^* \to [0, 1]$ is a computable function. The notation $P(w)$ for a string $w \in \Sigma^*$ stands for $P(C_w)$, the probability of the cylinder C_w.

Definition 4. A probability measure is called strongly positive if there is positive rational constant c ($0 < \mathrm{c} < 1$) such that

$$c < \inf_{\substack{w \in \Sigma^* \\ b \in \Sigma}} \{P(C_{wb} \mid C_w)\}. \tag{3}$$

We limit ourselves to the class of computable, strongly positive probability measures.

To define the notion of Martin-Löf random sequences, we introduce two concepts – that of a measure of impossibility, and that of a martingale.

Definition 5. A function $p : \Omega \to [0, \infty]$ is called a *measure of impossibility* with respect to the probability space $(\Omega, \mathcal{F}, P)$ if the following hold:

P1. p is lower semicomputable.

P2. $E_P\, p \leq 1$, where $E_P\, f$ is the expectation of the function f with respect to probability measure P.

A *measure of impossibility* p of ω with respect to the computable probability distribution P characterizes the "degree of disagreement" [23] of the outcome ω with respect to the given probability distribution. In particular, we can see that if ω is a not random with respect to the probability distribution P, then there is a P-measure of impossibility $p : \Omega \to [0, \infty]$ such that $p(\omega) = \infty$ [22].

Definition 6. Let $\omega \in \mathbf{C}$. Then ω is said to be P-impossible if there is a P-measure of impossibility p such that $p(\omega) = \infty$. A set $X \subseteq \mathbf{C}$ is said to be P-impossible, if

$$X \quad \subseteq \quad \{\omega : \omega \text{ is } P\text{-impossible}\}.$$

V'yugin and Vovk [22] proved that for every computable probability P, there is an optimal measure of impossibility $\tilde{p} : \mathbf{C} \to [0, \infty]$ such that a sequence ω is P-random if and only if $\tilde{p}(\omega) < \infty$. The set of Martin-Löf random sequences with respect to P is exactly the complement of the set of all $\omega \in \mathbf{C}$ that are P-impossible.

We also consider martingales.

Definition 7. A P-martingale $d : \Sigma^* \to [0, \infty]$, is a function which obeys the properties,

M1. $d(\lambda) \leq 1$.

M2. For all strings w, the following holds:

$$d(w)P(w) = d(w0)P(w0) + d(w1)P(w1).$$

A martingale is said to "succeed" on sequence ω if

$$\limsup_{n\to\infty} d(\omega[0\ldots n-1]) = \infty.$$

A martingale is said to "strongly succeed" on sequence ω if

$$\liminf_{n\to\infty} d(\omega[0\ldots n-1]) = \infty.$$

The success set of a martingale d, denoted $S^\infty[d]$, is defined to be the set of binary sequences on which d succeeds. The strong success set of a martingale d, denoted $S^\infty_{\text{str}}[d]$, is the set of binary sequences on which d strongly succeeds.

A constructive martingale is a lower semicomputable martingale.

We show, the concept of a measure of impossibility and that of a constructive martingale are equivalent, in that every measure of impossibility p corresponds to a martingale which wins on an ω if and only if $p(\omega) = \infty$. Since there is a universal martingale which succeeds on the set of non-random sequences, and there is a universal measure of impossibility which attains ∞ on the set of non-random sequences, it is indirectly known that there is a conversion between the success criteria of martingales and that of measures of impossibility. The new result here is a direct conversion of a martingale into a measure of impossibility and vice versa, such that the success sets of both are the same (under some assumptions on the probability measure).

4 Converting a Martingale into a Measure of Impossibility

Let P be a strongly positive computable probability measure. We wish to convert a lower semicomputable P-martingale which succeeds on a constructive P-measure-zero set, to a *measure of impossibility* $p : \mathbf{C} \to [0, \infty]$ with respect to P, such that $S^\infty[d]$ is P-impossible as witnessed by p. We show that if $d : \Sigma^* \to \mathbb{R}^+$ is a lower semicomputable P-martingale, then there exists a *measure of impossibility* $p : \Omega \to [0, \infty]$ such that $\forall \omega \in \Omega\ \limsup_{n\to\infty} d(\omega[0\ldots n-1]) = \infty$ if and only if $p(\omega) = \infty$.

We proceed in stages.

It is well-known that a sequence ω is non-random if and only if there is a martingale d which is such that $\liminf_n d(\omega[0\ldots n-1]) = \infty$. The following well-known lemma is stated here because we use the construction to prove results about the measure of impossibility.

Lemma 8. *[Schnorr [17]] If $d : \Sigma^* \to [0, \infty]$ is a constructive martingale, then there is a constructive martingale $d' : \Sigma^* \to [0, \infty]$ such that $S^\infty[d] \subseteq S^\infty_{str}[d']$. Moreover, there is a monotone function $\mathtt{sa} : \Sigma^* \to [0, \infty]$ such that $\lim_{n\to\infty} \mathtt{sa}(\omega[0 \ldots n-1]) = \infty$ if and only if $\omega \in S^\infty_{str}[d']$.*

Proof sketch:

The construction follows the construction of the martingale in Theorem 2.6 of [17]. Let the constant c witness the strong positivity of the computable probability measure P, as in Definition 4. For each n, let $A_n = \{w \in \Sigma^* \mid d(w) > [1/c]^n\}$. Let us denote, for any set of strings A, by $P(A)$, the probability $\cup_{w\in A} P(C_w)$. It follows that $P(A_n) \le c^n$. Since d is constructive, the A_n's are uniformly computably enumerable. For each computably enumerable set A_n, there is a recursive, prefix-free set B_n such that the set of infinite sequences with prefixes in B_n is exactly the same as those with prefixes in A_n.

For each n, the function $d_n : \Sigma^* \to \mathbb{R}^+$ is defined as

$$d_n(x) = \sum_{xy \in B_n} P(xy|x) + \mathtt{sa}_n(x)$$

where $\mathtt{sa}_n(x) = 1$ if some prefix of x is in B_n, and 0 otherwise. Then

$$d_n(\lambda) = \sum_{w \in B_n} P(w) \le c^{-n}.$$

Each d_n is a P-martingale, so $d' = \sum_{i=0}^\infty d_n$ is a P-martingale. The A_ns can be uniformly generated by an algorithm, hence so can the d_ns, it follows that d' is constructive.

Also, if $d(\omega[0 \ldots k-1]) > [1/c]^i$, then $\sum_{j=0}^i \mathtt{sa}_j(\omega[0 \ldots k-1]) > i$, so $d'(w[0 \ldots m-1]) > i$ for all $m > k$. This proves the lemma. $\square$

Let d' be a P-martingale as defined above. The measure of impossibility p is defined as follows.

$$p(\omega) = \lim_{n\to\infty} \sum_{i=0}^{n-1} \mathtt{sa}(\omega[0 \ldots n-1]). \tag{4}$$

We prove that p is a measure of impossibility which attains ∞ on all sequences on which d' strongly succeeds.

Lemma 9. *p defined in (4) is a measure of impossibility.*

Proof. Let $\hat{\mathtt{sa}}$ be the witness of the lower semicomputability of $\mathtt{sa}$. Then $\hat{\mathtt{sa}}$ witnesses that p is lower semicomputable.

Moreover,

$$\begin{aligned} E_P\, p &= \int p(\omega) dP \\ &\leq \liminf_{n\to\infty} \sum_{w'\in\{0,1\}^n} \int_{C_{w'}} \mathtt{sa}(w') dP \\ &= \lim_{n\to\infty} \sum_{w'\in\{0,1\}^n} \int_{C_{w'}} \mathtt{sa}(w') dP, \end{aligned}$$

by Fatou's Lemma. But, since $\mathtt{sa}(w') \leq d(w')$ for all w', it follows by Kraft's inequality that

$$\sum_{w'\in\{0,1\}^n} \int \mathtt{sa}(w') P(w') dP \leq \sum_{w'\in\{0,1\}^n} \int d'(w') P(w') dP \leq 1,$$

for all n. Hence it follows that $\int p(\omega) dP \leq 1$. Therefore p defines a *measure of impossibility.* □

Let $\omega \in S^\infty[d]$. It is clear that, since $\liminf_{n\to\infty} d'(\omega[0\ldots n-1]) = \infty$ implies that $\sup_n \mathtt{sa}(\omega[0\ldots n-1]) = \infty$, we have $p(\omega) = \infty$. Thus p satisfies the conditions of being a measure of impossibility which attains ∞ on ω.

5 Converting a Measure of Impossibility into a Martingale

We assume $P : \mathcal{F} \to [0,1]$ is a computable probability measure. If $p : \Omega \to [0,\infty]$ is a *measure of impossibility* with respect to P, we prove: there exists a constructive P-martingale $d : \Sigma^* \to \mathbb{R}^+$ such that d succeeds on every ω on which p assumes ∞– *i.e.* $\{\omega : \limsup_{n\to\infty} d(\omega[0\ldots n-1]) = \infty\} \supseteq \{\omega : p(\omega) = \infty\}$, with equality if P is a measure which assigns positive probability to every cylinder.

We make the following restrictions: We ensure that P is not just a computable probability measure, but also *very strongly* positive: if $\hat{P}$ testifies to the fact that P is computable, then there exists a program $f : \Sigma^* \times 0^{\mathbb{N}} \to \mathbb{Q}$ such that for every cylinder C_x, the probability of the cylinder $P(C_x) > 0$ if and only if for all positive integers n, we have $\hat{P}(x, 0^n) > f(x, 0^n)$. Note that if a probability measure is strongly positive, then it is very strongly positive.

We define the P-martingale d.

For the empty string, $d(\lambda) = E_P[p]$. For all strings w, and $b \in \{0,1\}$,

$$d(wb) = \begin{cases} E_P[p \mid C_{wb}] & \text{if } P(C_{wb}) > 0 \\ 2 \cdot d(w) & \text{otherwise.} \end{cases} \tag{5}$$

Lemma 10. *If p is a measure of impossibility with respect to a very strongly positive, computable probability measure P, then d defined in (5), is a lower semicomputable P-martingale.*

Proof. We show that d is a P-martingale, and d is lower semicomputable,

d is a P-martingale:

We have $d(\lambda) = \frac{\int_\Omega p(\omega)dP}{P(\Omega)} \leq 1$.

For a string w, if all of C_w, C_{w0} and C_{w1} have non-zero probability, then the stipulation (M2) is satisfied by linearity of conditional expectation. If, $P(C_w) \neq 0$, but one of $P(C_{w0})$, $P(C_{w1})$ is zero, without loss of generality, say $P(C_{w0}) = 0$, then

$$\begin{aligned} d(w0)P(C_{w0}) + d(w1)P(C_{w1}) &= 2d(w) \times 0 + E[p|C_{w1}]P[C_{w1}] \\ &= E[p|C_{w1}]P(C_w) = E[p|C_w]P(C_w) \\ &= d(w)P(C_w). \end{aligned}$$

d is lower semicomputable:

Consider the following program: Algorithm for $\hat{d} : \Sigma^* \times 0^{\mathbb{N}} \to \mathbb{Q}$: Let $x \in \Sigma^*, b \in \Sigma$.

procedure $\hat{d}(xb, 0^n)$
  Input $xb \in \Sigma^*(b \in \Sigma)$ and $n \in \mathbb{N}$.
  if $f(xb, 0^n) > \hat{P}(xb, 0^n)$ **then**
    $\hat{d}(xb, 0^n) = 2\,\hat{d}(x, 0^n)$.
  else

$$\hat{d}(xb, 0^n) = \frac{\sum_{w \in \{0,1\}^n} \max_{y \sqsubseteq xbw}\{\hat{p}(y, 0^n)\} \times (\hat{P}(xbw, 0^{2n+1}) - 2 \cdot 2^{-2n-1})}{P(xb, 0^{2n+1}) + 2 \cdot 2^{-2n-1}}.$$

  end if
end procedure

To show that $\hat{d}(xb, \cdot)$ is a lower semicomputation of $d(xb)$, we proceed as follows. We prove that the numerator in else statement converges to the appropriate limit. From this, it follows that the output of the program converges to the value of d for the given string xb from below in a lower semicomputable way.

Define the following:

$$\forall xb \in \Sigma^*, m \in \mathbb{N}\ f_m^{xb0^\infty} = \sum_{w \in \{0,1\}^m} \max_{y \sqsubseteq xbw} \{\hat{p}(y, 0^m)\} \left[\hat{P}(xbw, 0^{2m+1}) - 2 \cdot 2^{-2m-1}\right]$$

$$\forall xb \in \Sigma^*, m \in \mathbb{N}\ S_m^{xb0^\infty} = \sum_{w \in \{0,1\}^m} \max_{y \sqsubseteq xbw} \{\hat{p}(y, 0^m)\} P(xbw)$$

The following claims suffice to prove that $\hat{d} : \mathbb{N} \times \Sigma^* \rightarrow \mathbb{Q}$ is a lower semicomputation of d.

Lemma 11. $\forall m \in \mathbb{N}\ f_m^{xb0^\infty} \leq S_m^{xb0^\infty} \leq \int_{C_{xb}} p(\omega)\, dP$.

Proof sketch: Lower semicomputability of P with witness $\hat{P}(\cdot, 0^m) - 2.2^{-m}$ for all $m \in \mathbb{N}$ implies the first inequality. The second is by the lower semicomputability of $\hat{p}$ with respect to p, and the fact that each $S_m^{x0^\infty}$ is the integral of a step function defined on Ω, $\hat{p} < p$ (everywhere), and by the definition of the Lebesgue integral. □

Now, we show that the sum converges as $n \rightarrow \infty$ to the required integral:

Lemma 12. *The series $f_m^{xb0^\infty}$ converges uniformly to the same limit as of the sum series $S_m^{xb0^\infty}$ as $m \rightarrow \infty$.*

Proof. By the computability witness $\hat{P}$ of P, we have, for any xbw, $m \in \mathbb{N}$,

$$P(xbw) - \hat{P}(xbw, 0^{2(m+1)}) < \frac{1}{2^{(2m+1)}},$$

whereby

$$|f_m^{xb0^\infty} - S_m^{xb0^\infty}| < \frac{1}{2^{2m+1}} \times 2^m = \frac{1}{2^{m+1}}$$

□

The fact that $S_m^{xb0^\infty} \rightarrow \int_{C_{xb}} p(\omega) dP$ as $m \rightarrow \infty$, follows due to the fact that $\hat{p}$ is a lower semicomputation of p. Property (1) of lower semicomputability ensures that p dominates the step function summed in $S_m^{xb0^\infty}$. The convergence property of lower semicomputability ensures that the the function $S_m^{xb0^\infty}$ converges to the integral $\int pdP$.

These claims suffice to establish the condition that $\hat{d}$ has to satisfy to be a lower semicomputation of d.

□

Lemma 13. *For any $\omega \in \mathbf{C}$, $p(\omega) = \infty$ implies* $\limsup_{n\to\infty} d(\omega[0 \to n-1]) = \infty$.

Proof. First, if $P(C_{\omega[0...n-1]}) = 0$ for some n, this is routine to prove.

So, assume for all n, $P(C_{\omega[0...n-1]}) > 0$, so that for all n, $d(\omega[0 \ldots n-1]) = E[p|C_{\omega[0...n-1]}]$. We show that d succeeds on ω. It is enough to show that for every rational q, there is some $x \sqsubseteq \omega$ such that $d(x) > q$.

Let S be the graph of p, described by the union of a computable enumeration $\mathcal{C}$ of cylinders in $\mathbf{C} \times \mathbb{Q}$. If $p(\omega) = \infty$, then for every $q \in \mathbb{Q}$, there is an $(x, q) \in \mathcal{C}$ such that x is some prefix of ω. If this is so, then for every $\sigma \in C_x$, $p(\sigma) > q$, whence $E[p|C_x] > q$, which proves the result. □

6 A New Characterization of Constructive Dimension

In this section, we generalize the construction of the previous sections, to come up with an alternate definition of constructive Hausdorff and constructive packing dimension.

For $s \in [0, \infty)$, we introduce the notion of a set being s-improbable with respect to a measure of impossibility.

Definition 14. Let $X \subseteq \mathbf{C}$. We say that X is *s-improbable with respect to a P-measure of impossibility* $p : \mathbf{C} \to [0, \infty]$ if for every infinite binary sequence $\omega \in X$, we have

$$\limsup_{n\to\infty} \frac{\int_{C_{\omega[0...n-1]}} p(\omega) dP}{P^s(C_{\omega[0...n-1]})} = \infty. \tag{6}$$

X is *strongly s-improbable with respect to p* if

$$\liminf_{n\to\infty} \frac{\int_{C_{\omega[0...n-1]}} p(\omega) dP}{P^s(C_{\omega[0...n-1]})} = \infty. \tag{7}$$

The concept of s-improbability generalizes the concept of improbability.

Lemma 15. *Let $p : \mathbf{C} \to [0, \infty]$ be a P-measure of impossibility.*

1. *For any $\omega \in \mathbf{C}$, if $p(\omega) = \infty$, then* $\limsup_{n\to\infty} \frac{\int_{C_{\omega[0...n-1]}} p(\omega) dP}{P(C_{\omega[0...n-1]})} = \infty$.
2. *For any $\omega \in \mathbf{C}$, if* $\limsup_{n\to\infty} \frac{\int_{C_{\omega[0...n-1]}} p(\omega) dP}{P(C_{\omega[0...n-1]})} = \infty$, *then there exists a P-measure of impossibility $p' : \mathbf{C} \to [0, \infty]$ such that $p'(\omega) = \infty$.*

Proof. 1.) Let $\omega \in \mathbf{C}$ be such that $p(\omega) = \infty$. Since p is lower semicomputable and $P(C_w) > 0$ for all strings w, the function $d : \Sigma^* \to \mathbb{R}^+$ defined by $d(w) = E_P[p|C_w]$ is a martingale, such that $\omega \in S^\infty[d]$. Thus, $\limsup_{n\to\infty} \frac{\int_{C_{\omega[0\ldots n-1]}} p(\omega)dP}{P(C_{\omega[0\ldots n-1]})} = \infty$.

2.) Since $\int pdP \leq 1$ and $P(C_w) > 0$ for all w, we take the equivalent characterization that $\sup_n \frac{\int_{C_{\omega[0\ldots n-1]}} p(\omega)dP}{P(C_{\omega[0\ldots n-1]})} = \infty$. Since p is lowersemicomputable and P is computable, $f' : \mathbf{C} \to [0,\infty]$ defined by $f'(\chi) = \sup_n \frac{\int_{C_{\chi[0\ldots n-1]}} p(\chi)dP}{P(C_{\chi[0\ldots n-1]})} = \infty$ is a measure of impossibility. □

We now review the notion of a lower semicomputable s-P-gale, which, following Lutz [14], we use to give a definition of constructive Hausdorff (or constructive Billingsley) dimension.

Definition 16. (Lutz [14]) Let $s \in [0,\infty)$. An *s-P-gale* $d : \Sigma^* \to \mathbb{R}^+$ is a function that satisfies the condition for all $w \in \Sigma^*$,

$$d(w)P^s(w) \quad = \quad [d(w0)P^s(w0) + d(w1)P^s(w1)] \tag{8}$$

Definition 17. (Lutz [14]) Let d be an s-P-gale, where $s \in [0,\infty)$.

- We say that d succeeds on $\omega \in \mathbf{C}$ if
$$\limsup_{n\to\infty} d(\omega[0\ldots n-1]) = \infty.$$
- The success set of d is
$$S^\infty[d] = \{\omega \in \mathbf{C} \mid d \text{ succeeds on } \omega\}.$$
- We say that d strongly succeeds on $\omega \in \mathbf{C}$ if
$$\liminf_{n\to\infty} d(\omega[0\ldots n-1]) = \infty.$$
- The strong success set of d is
$$S^\infty_{str}[d] = \{\omega \in \mathbf{C} \mid d \text{ strongly succeeds on } \omega\}.$$

The notion of constructive Hausdorff dimension, a constructive analogue of the classical Hausdorff dimension, is defined using the notion of constructive s-gales [14].

Remark.[Lutz [14]] For every $s_1, s_2 \in [0,\infty)$, the function $d : \Sigma^* \to [0,\infty]$ is a P-s_1-gale if and only if the function $d' : \Sigma^* \to [0,\infty]$ defined by $d'(w) = P^{(s_1-s_2)}(w)d(w)$ is an s_2 gale.

Definition 18. (Lutz [14], Lutz and Mayordomo [10]) The constructive *Hausdorff dimension* of a set $X \subseteq \mathbf{C}$ is

$$\dim_H^P(X) \quad = \quad \inf\{s \in [0,\infty) \mid \text{ There is a constructive } s\text{-}P\text{-gale for which } X \subseteq S^\infty[d].\}$$

Analogously, the notion of constructive packing dimension is defined as the constructive analogue of the classical packing dimension [20], [19]. We use the following equivalent notion defined using strong success of s-P-gales.

Definition 19. (Athreya et al. [1], [10]) The constructive *packing dimension* of a set $X \subseteq \mathbf{C}$ is

$$\mathrm{Dim}_H^P(X) \quad = \quad \inf\{s \in [0,\infty) \mid \text{ There is a constructive } s\text{-}P\text{-gale for which } X \subseteq S^\infty_{\mathrm{str}}[d].\}$$

Notation. For $X \subseteq \mathbf{C}$, let $\mathcal{G}(X)$ be the set of all $s \in [0,\infty)$ such that there is an s-P-gale d for which $X \subseteq S^\infty[d]$ and $\mathcal{P}(X)$ be the set of all $s \in [0,\infty)$ such that X is s-improbable with respect to some P-measure of impossibility. Similarly, let $\mathcal{G}_{\mathrm{str}}(X)$ be the set of all $s \in [0,\infty)$ such that there is an s-P-gale d for which $X \subseteq S^\infty_{\mathrm{str}}[d]$ and $\mathcal{P}_{\mathrm{str}}(X)$ be the set of all $s \in [0,\infty)$ such that X is strongly s-improbable with respect to some P-measure of impossibility.

The following lemma asserts that for every strongly positive computable probability measure P, every positive rational s, for every $X \subseteq \mathbf{C}$, there is a P-s-gale which succeeds on X if and only if X is s-improbable with respect to the probability measure, and analogously, there is an P-s-gale which strongly succeeds on X if and only if X is strongly s-improbable with respect to the probability measure.

The construction uses an analogue of the "savings account" method used in the proof of Lemma 7. In Lemma 7, the method was used to prove that the set of sequences on which there is a P-martingale succeeds, is the same as that on which there is a P-martingale which strongly succeeds. In the case of s-gales, we cannot prove this: there are sequences and some $s \in [0,\infty)$ such that there is some constructive s-gale which succeeds on them, but there are no s-gales which succeed strongly. However, we used the construction in Lemma 7 to extract a monotone behavior from the martingale, which was used in the definition of the measure of impossibility. The construction in the next lemma shows that it is possible to extract a monotone behavior from an s-gale to define a measure of impossibility which has desirable properties.

Lemma 20. *Let P be a strongly positive computable probability measure. Let $X \subseteq \mathbf{C}$ and $s \in [0,\infty)$ be a rational. Then $s \in \mathcal{G}(X)$ if and only if $s \in \mathcal{P}(X)$, and $s \in \mathcal{G}_{str}(X)$ if and only if $s \in \mathcal{P}_{str}(X)$.*

Figure 1: Algorithm 2

procedure $d^{(n)}(w)$
  $\alpha_n = 2c^{(s_2-s_1)n/2}$
  if $|w| < n$ **then**
    $\mathtt{bc}'_n(w) \leftarrow \alpha_n d_2(w)$
    $\mathtt{sa}'_n(w) \leftarrow 0$
  else
    $val_w \leftarrow d_2(w[0 \ldots n-1])P^{(1-s_2)}(C_w | C_{w[0\ldots n-1]})$
    if $d_2(w[0 \ldots n-1]) > P^{s_1-s_2}(C_{w[0\ldots n-1]})$ **then**
      $\mathtt{bc}'_n(w), \mathtt{sa}'_n(w) \leftarrow \frac{\alpha_n val_w}{2}$
    else
      $\mathtt{bc}'_n(w) \leftarrow \alpha_n val_w$
      $\mathtt{sa}'_n(w) \leftarrow 0$
    end if
  end if
  return $\mathtt{bc}'_n(w) + \mathtt{sa}'_n(w)$
end procedure

Proof. Let c be as in (3). Let $\epsilon > 0$ be arbitrary and $s_1 < 1$ be rational, and let d_1 be an s_1-P-gale. Let s_2 be a rational number such that $s_2 \in (s_1, \min\{1, s_1 + \epsilon\})$. Then $d_2 : \Sigma^* \to R$ defined by, for all strings w, $d_2(w) = d_1(w)P(w)^{(s_1-s_2)}$ is an s_2-P-gale such that $S^\infty[d_1] \subseteq S^\infty[d_2]$ and $S^\infty_{\text{str}}[d_1] \subseteq S^\infty_{\text{str}}[d_2]$. As in the case of constructive martingales, we can form another s_2-P-gale d' consisting of $\mathtt{bc}'$ and $\mathtt{sa}'$ such that the following hold.

1. For all strings w, $d'(w) = \mathtt{bc}'(w) + \mathtt{sa}'(w)$.
2. $\mathtt{bc}'(\lambda) = d_2(\lambda)$.
3. $\mathtt{sa}'(\lambda) = 0$.

The construction is an adaptation of the construction in [3, 5]. Using this, we can build a P-measure of impossibility p which witnesses that X is s_2-improbable wrt P.

Consider $d^{(n)} : \Sigma^* \to \mathbb{Q}$ defined in Algorithm 2.

In Algorithm 2, we have, for every string $w \in \{0,1\}^{<n}$, $d^{(n)}$ behaves exactly the same as $\alpha_n d_2$.

For $wb \in \Sigma^{=n}$, we have $val_{wb} = d_2(wb)$. We have that half of the betting capital at wb is transferred to the the savings account of wb if $d_2(wb) >$

$P^{(s_1-s_2)}(wb)$. This transfer maintains the property $\sum_{b\in\{0,1\}} d^{(n)}(wb)P^{s_2}(wb) = d^{(n)}(w)P^{s_2}(w)$. This transfer preserves the s_2-P-gale property of $d^{(n)}$.

Also, for $w \in \{0,1\}^{\geq n}$ and $b \in \{0,1\}$, in lines 8–13,

$$\begin{aligned}\sum_{b\in\{0,1\}} \mathtt{bc}'_n(w)P^{1-s_2}(C_{wb}|C_w)P^{s_2}(C_{wb}) =& \mathtt{bc}'_n(w)\frac{P(C_w)}{P^{1-s_2}C_w} \\ =& \mathtt{bc}'_n(w)P^{s_2}(C_w),\end{aligned}$$

and

$$\begin{aligned}\sum_{b\in\{0,1\}} \mathtt{sa}'_n(w)\left[\frac{P^{1-s_2}(C_{wb})}{P^{1-s_2}(C_w)}\right]P^{s_2}(C_{wb}) &= \sum_{b\in\{0,1\}} \mathtt{sa}'_n(w)\left[\frac{P(C_{wb})}{P^{1-s_2}(C_w)}\right] \\ &= \mathtt{sa}'_n(w)P^{s_2}(C_w),\end{aligned}$$

so the s_2-P-gale condition for $d^{(n)}$ is satisfied. Hence, each $d^{(n)}$ is an s_2-P-gale.

Denote $\alpha = \sum_{n=0}^{\infty} \alpha_n$. This is finite since $c < 1$. Define $d' = 1/\alpha \sum_{i=0}^{\infty} d^{(n)}$. Then $d'(\lambda) = d_2(\lambda)$. It follows that d' is an s_2-P-gale.

We note that P is a computable probability measure, s_2 is a rational number and d_2 is a constructive s_2-P-gale. For each n, the condition in line 8 can be verified by a lower semicomputation. It follows that each d_n is a constructive s_2-P-gale, and hence, so is d'.

We define $\mathtt{bc}'(w) = \sum_{i=0}^{|w|-1} \mathtt{bc}'_i(w)$. Let $\mathtt{sa}'(w) = \sum_{i=0}^{|w|-1} \mathtt{sa}'_i(w)$. Each $\mathtt{sa}_n$ is lower semicomputable by a computation that is 0 until the condition in line 8 is true, and converges from below to $\frac{\alpha_n val_w}{2}$ when the condition is true. Hence $\mathtt{sa}'$ is also lower semicomputable. Each $\mathtt{sa}_n$ is lower semicomputable by a computation that is 0 until the condition in line 8 is true, and converges from below to $\frac{\alpha_n val_w}{2}$ when the condition is true. Hence $\mathtt{sa}'$ is also lower semicomputable. Unlike the martingale case, we cannot say that $\mathtt{sa}'(w)$ is monotone increasing in the length of w. However, we can use the remark previously noted, to construct a martingale which for every w is defined as $\mathtt{bc}'(w)P^{s_2-1}(C_w) + \mathtt{sa}'(w)P^{s_2-1}(C_w)$.

We prove that for each n, $\mathtt{sa}'_n(w)P^{(s_2-1)}(C_w)$ is constant for all w with at least n bits. To see this, let w be a string such that $|w| \geq n$. If $d_2(w[0\ldots n-$

$1]) > P^{s_1-s_2}(C_{w[0\ldots n-1]})$, then we have

$$\begin{aligned}\mathsf{sa}'_n(w)P(C_w) &= \frac{\alpha_n}{2} val_w P^{(s_2-1)}(C_w) \\ &= \frac{\alpha_n}{2} d_2(w[0\ldots n-1])P^{(1-s_2)}(C_w|C_{w[0\ldots n-1]})P^{(s_2-1)}(C_w) \\ &= \frac{\alpha_n}{2} d_2(w[0\ldots n-1])\frac{1}{P^{(1-s_2)}(C_{w[0\ldots n-1]})},\end{aligned}$$

which is a constant. If, $d_2(w[0\ldots n-1]) \leq P^{s_1-s_2}(C_{w[0\ldots n-1]})$, then sa'_n is zero, so $\mathsf{sa}'_n(w)P^{(s_2-1)}(C_w)$ is zero.

We can see that $\mathsf{sa}'(w)P^{s_2-1}(C_w)$ is monotone increasing with the length of w. This is because for each i, sa'_i appears in the summation of sa' only on strings which are at least i bits long.

We define $\overline{p} : \Sigma^* \to \mathbb{R}^+$ by

$$\overline{p}(w) = \mathsf{sa}'(w)P^{s_2-1}(C_w),$$

for finite strings w.

Now, we define $p : \mathbf{C} \to [0,\infty]$ by

$$p(\omega) = \lim_{n\to\infty} \overline{p}(\omega[0\ldots n-1]).$$

Since sa' is lower semicomputable, it follows that p is lower semicomputable. Then we have $\int_{C_\lambda} pdP \leq d_2(\lambda)P(\mathbf{C}) = 1$. Thus p is a measure of impossibility.

Also,

$$\frac{\int_{C_w} p(\omega)dP}{P^{s_2}(C_w)} \geq \mathsf{sa}'(w)$$

for all strings w.

We have to analyze the behavior of sa' on sequences where d_1 succeeds. We can observe the following fact. If the s_1-gale d_1 succeeds on ω, then for every i, there is a least n such that $d_1(\omega[0\ldots n-1]) \geq 2[1/c]^i$. Then, at this length n, $d_2(\omega[0\ldots n-1]) \geq 2[1/c]^i P(\omega[0\ldots n-1])^{(s_1-s_2)}$. This quantity is at least $[1/c]^{i+(s_2-s_1)n}$. Then $\mathsf{sa}'_n(\omega[0\ldots n-1]) > [1/c]^{i+(s_2-s_1)n/2}$, so $\mathsf{sa}'(\omega[0\ldots n-1]) > [1/c]^{\Omega(n)}$.

So we have that

$$\limsup_{n\to\infty} d_1(\omega[0\ldots n-1]) = \infty \Leftrightarrow \limsup_{n\to\infty} \mathsf{sa}'(\omega[0\ldots n-1]) = \infty.$$

If d_1 strongly succeeds on ω, then for each i, for all large enough n, we have $d_1(\omega[0\dots n-1]) > 2[1/c]^i$, which implies $d_2(\omega[0\dots n-1]) \geq 2[1/c]^i P(\omega[0\dots n-1])^{(s_1-s_2)}$, and hence $\mathsf{sa}'_n(\omega[0\dots n-1]) > [1/c]^{i+(s_2-s_1)n-0.5(s_2-s_1)n} = [1/c]^{\Omega(n)}$. This implies that, $\liminf_{n\to\infty} \mathsf{sa}'(\omega[0\dots n-1]) = \infty$.

Thus,

$$\liminf_{n\to\infty} d_1(\omega[0\dots n-1]) = \infty \Leftrightarrow \liminf_{n\to\infty} \mathsf{sa}'(\omega[0\dots n-1]) = \infty.$$

Hence

$$\limsup_{n\to\infty} d(\omega[0\dots n-1]) = \infty \quad \Longrightarrow \quad \limsup_{n\to\infty} \frac{\int_{C_{\omega[0\dots n-1]}} p(\omega)dP}{P^{s_2}(C_{\omega[0\dots n-1]})} = \infty.$$

Similarly,

$$\liminf_{n\to\infty} d(\omega[0\dots n-1]) = \infty \quad \Longrightarrow \quad \liminf_{n\to\infty} \frac{\int_{C_{\omega[0\dots n-1]}} p(\omega)dP}{P^{s_2}(C_{\omega[0\dots n-1]})} = \infty.$$

Thus $s_2 \in \mathcal{P}_{\text{str}}(X)$ if $s_2 \in \mathcal{G}_{\text{str}}(X)$, and $s_2 \in \mathcal{P}(X)$ if $s_2 \in \mathcal{G}(X)$.

If $s = 1$, the condition that $\limsup_{n\to\infty} \frac{\int_{C(\omega[0\dots n-1])} pdP}{P^s(C_{\omega[0\dots n-1]})} = \limsup_{n\to\infty} E_P[p|C_{\omega[0\dots n-1]}]$ is similar to the case of conversion of the martingale case.

If $s > 1$, and let $\epsilon > 0$ be arbitrary. then we pick a rational number s_2 such that $s < s_2 < s + \epsilon$. The s_2-gale d_2 which bets, for every string w and every symbol b, in the proportion $P(C_{wb}|C_w)^{1-s_2}$ succeeds on the Cantor Space $\mathbf{C}$. We define $\overline{p}(\omega[0\dots n-1]) = d_2(\omega[0\dots n-1])P^{s_2-1}(\omega[0\dots n-1])$, a monotone function in n. Then $p(\omega) = \lim_{n\to\infty} p(\omega[0\dots n-1])$ is a measure of impossibility. Then

$$\frac{\int_{\omega[0\dots n-1]} pdP}{P^{s_2}(C_{\omega[0\dots n-1]})} = d_2(\omega[0\dots n-1]),$$

so ω is s_2-improbable with respect to P if $\omega \in S^\infty[d_2]$, and ω is strongly s_2-improbable with respect to P if $\omega \in S^\infty_{\text{str}}[d_2]$.

Conversely, let $s \in \mathcal{P}(X)$. Then there exists a measure of impossibility p such that X is s-improbable with respect to p. We define a martingale $d : \Sigma^* \to \mathbb{R}^+$ as follows. For finite strings w,

$$d(w) \quad = \quad E_P[p|C_w]P^{1-s}(C_w).$$

It is routine to see that d is a lowersemicomputable s-P-gale. Moreover,

$$\limsup_{n\to\infty} \frac{\int_{C_{\omega[0\dots n-1]}} pdP}{P^s(C_{\omega[0\dots n-1]})} = \infty \quad \Longrightarrow \limsup_{n\to\infty} d(\omega[0\dots n-1]) = \infty.$$

Thus, we can see that $s \in \mathcal{G}(X)$. Similarly, we can establish that if $s \in \mathcal{P}_{\mathrm{str}}(X)$, then $s \in \mathcal{G}_{\mathrm{str}}(X)$.

□

Using this, we can characterize effective Hausdorff and packing dimensions in the following way.

Corollary 21. *(Alternate Characterization of Constructive Dimension) For any set $X \subseteq \mathbf{C}$, the constructive* Hausdorff P-dimension *of X $= \inf \mathcal{P}(X)$, and the* constructive P-packing *dimension of X $= \inf \mathcal{P}_{str}(X)$.*

7 Scaled Dimension

In this section, we give an alternate characterization of constructive scaled dimension, introduced by Hitchcock, Lutz and Mayordomo [4]. The discussion in this section focuses on Lebesgue measure, since the theory of scaled dimension has been developed so far for the Lebesgue measure (or equivalently, uniform probability distribution).

Scaled dimension is a generalization of classical Hausdorff and classical packing dimension. Just as Hausdorff dimension and packing dimension allows us to make distinctions between measure 0 sets, scaled dimension allows us to make distinctions between dimension 0 sets. The motivation for studying scaled dimension comes from complexity theory and its applications, like cryptography, where many interesting classes have resource-bounded dimension 0 in the appropriate resource-bounded setting (for example, ESPACE) [6].

However, we give a characteracterization of scaled dimension at the *constructive* level, using a suitable generalization of the characterization for constructive dimension, given in the previous section. We first give the notion of a scale, and then introduce the notion of a scaled improbability, and use this notion to give a new characterization of scaled dimension.

Definition 22. (Hitchcock, Lutz, Mayordomo, 2004 [4]) A *scale* is a continuous function $g : H \times \mathbb{R} \to \mathbb{R}$ with the following properties.

1. $H = (a, \infty)$ for some $a \in \mathbb{R} \cup \{\infty\}$.
2. $g(m, 1) = m$ for all $m \in H$.
3. $g(m, 0) = g(m', 0) \geq 0$ for all $m, m' \in H$.
4. For every sufficiently large $m \in H$, the function $s \mapsto g(m, s)$ is nonnegative and strictly increasing.

5. For all $s' > s \geq 0$, $\lim_{m\to\infty}[g(m,s') - g(m,s)] = \infty$.

Notation Let g be a scale. Then for any $j \in H$, we define

$$\delta_j(s_1, s_2) \;=\; 2^{g(j,s_2)-g(j,s_1)}.$$

If $s_1 < s_2$, then $\delta_j(s_1, s_2) > 1$.

We deal with computable rapidly growing scales in this section. We give the definition of a scaled gale.

Definition 23. (Hitchcock, Lutz and Mayordomo [4]) Let $g : H \times \mathbb{R} \to \mathbb{R}$ be a computable scale, and $s \in [0, \infty)$.

A g-scaled s-gale (briefly, an $s^{(g)}$*-gale*) is a function $d : \Sigma^* \to [0, \infty)$ such that for all $w \in \Sigma^*$ with $|w| \in H$,

$$d(w)2^{-g(|w|,s)} \;=\; d(w0)2^{-g(|w0|,s)} + d(w1)2^{-g(|w1|,s)}. \tag{9}$$

An s-gale (whose definition is given in section 6) is a g-scaled s-gale where $g(n, s) = ns$ for all s and n.

The notion of success and strong success of a g-scaled s-gale are exactly the lim sup and the lim inf success defined in section 6. Therefore, we do not reproduce it here.

We now introduce a notion of scaled improbability, based on the notion of the scale functions defined above.

Definition 24. Let $X \subseteq \mathbf{C}$. We say that X is $s^{(g)}$*-improbable* with respect to a measure of impossibility[1] $p : \mathbf{C} \to \mathbb{R}^+$ if for every binary sequence $\omega \in X$, we have

$$\limsup_{n\to\infty} \frac{\int_{C\omega[0\ldots n-1]} p(\omega)d\omega}{2^{-g(n,s)}} \;=\; \infty. \tag{10}$$

We say X is *strongly* $s^{(g)}$*-improbable* with respect to p if for every binary sequence $\omega \in X$, we have

$$\liminf_{n\to\infty} \frac{\int_{C\omega[0\ldots n-1]} p(\omega)d\omega}{2^{-g(n,s)}} \;=\; \infty. \tag{11}$$

We introduce some notation which is used in the definition of scaled dimension. We recall the standard definitions of scaled dimension, and scaled strong dimension, and then give an alternate characterization of scaled dimension using the concept of scaled improbability.

[1]The probability measure is the uniform probability measure.

Notation. For $X \subseteq \mathbf{C}$, let $\mathcal{G}^{(g)}(X)$ be the set of all $s \in [0,\infty)$ such that there is an $s^{(g)}$-gale d for which $X \subseteq S^{\infty}[d]$ and $\mathcal{P}(X)$ be the set of all $s \in [0,\infty)$ such that X is $s^{(g)}$-improbable with respect to some measure of impossibility. Similarly, let $\mathcal{G}^{(g)}_{\mathrm{str}}(X)$ be the set of all $s \in [0,\infty)$ such that there is an $s^{(g)}$-gale d for which $X \subseteq S^{\infty}_{\mathrm{str}}[d]$ and $\mathcal{P}_{\mathrm{str}}(X)$ be the set of all $s \in [0,\infty)$ such that X is strongly $s^{(g)}$-improbable with respect to some P-measure of impossibility.

Definition 25. (Hitchcock, Lutz, Mayordomo [4]) If g is a scale, then the g-scaled dimension of a set $X \subseteq \mathbf{C}$ is $\dim^{(g)}(X) = \inf \mathcal{G}^{(g)}(X)$. The g-scaled strong dimension of X is $\mathrm{Dim}^{(g)}(X) = \inf \mathcal{G}^{(g)}(X)$.

We introduce the notion of a rapidly growing scale.

Definition 26. Let p be the smallest natural number in H. A rapidly growing scale is a scale such that for any s, s' where $s' > s$, there is a constant $0 < k < 1$ such that $\sum_{m=p}^{\infty} 2^{-k\delta_m(s',s)}$ is finite.

The above notion is reminiscent of the notion of strong positivity of the probability measure dealt with in the preceding sections, since strong positivity also ensures convergence of an infinite series necessary in the construction of the countable family of martingales.

The following lemma is crucial in the alternate characterization of scaled dimension. The constructions we use in the proof of the lemma are exactly analogous to those in the proof of Lemma 20 (in fact, in the case of the uniform probability measure, the constructions in Lemma 20 are special cases of the following constructions when the scale is $g(n,s) = ns$.).

Lemma 27. *Let g be a computable rapidly growing scale, $s \in [0,\infty)$ and $X \subseteq \mathbf{C}$. Then $s \in \mathcal{G}^{(g)}(X)$ if and only if $s \in \mathcal{P}^{(g)}(X)$, and $s \in \mathcal{G}^{(g)}_{str}(X)$ if and only if $s \in \mathcal{P}^{(g)}_{str}(X)$.*

Proof. Let g be a computable rapidly growing scale.

Let $s_1 \in [0,\infty)$ be a rational such that $g(n, s_1) < n$ for all $n \in \mathbb{N}$, when defined, and $\epsilon > 0$ be arbitrary. Let $s_2 \in [0,\infty)$ be such that for every n where both are defined, $\epsilon f(n) < g(n,s_2) - g(n,s_1)$, and $g(n,s_2) < n$. Let p be the smallest natural number such that both $g(p,s_1)$ and $g(p,s_2)$ are defined, and $M > 2^{-g(p,s_2)-g(p,s_1)}$. Let $0 < k < 1$ be such that $\sum_{m=p}^{\infty} 2^{-k\delta_n(s_2,s_1)} < \infty$. Let $d_1 : \Sigma^* \to \mathbb{R}$ be a g-scaled s_1-gale. Then $d_2 : \{0,1\}^* \to \mathbb{R}$ defined by $d_2(w) = d_1(w) 2^{g(|w|,s_2)-g(|w|,s_1)}$ is a g-scaled s_2-gale.

Figure 2: Algorithm 3

procedure $d^{(n)}(w)$
  $\alpha_n = 2^{-k\delta_n(s_2,s_1)}$
  if $g(n, s_2)$ is not defined **then**
    $\mathtt{bc}^{(n)'}(w) \leftarrow \alpha_n d_2(w)$
    $\mathtt{sa}^{(n)'}(w) \leftarrow 0$
    return $\mathtt{bc}^{(n)'}(w) + \mathtt{sa}^{(n)'}(w)$
  end if
  if $|w| < n$ **then**
    $\mathtt{bc}^{(n)'}(w) \leftarrow \alpha_n d_2(w)$
    $\mathtt{sa}^{(n)'}(w) \leftarrow 0.$
  else
    $val_w = d_2(w[0 \ldots n-1])2^{g(j,s_2)-g(j-1,s_2)-1}$
    if $d_2(w[0 \ldots n-1]) > 2^{\delta_n(s_2,s_1)}$ **then**
      $\mathtt{bc}^{(n)'}(w), \mathtt{sa}^{(n)'}(w) \leftarrow \alpha_n val_w/2$
    else
      $\mathtt{bc}^{(n)'}(w) \leftarrow \alpha_n val_w$
      $\mathtt{sa}^{(n)'}(w) \leftarrow 0$
    end if
  end if
  return $\mathtt{bc}^{(n)'}(w) + \mathtt{sa}^{(n)'}(w)$
end procedure

We will construct the following g-scaled s_2-gale, which transfers some capital into a "savings account". We will use this gale to build the $s_2^{(g)}$-measure of improbability. The gale $d^{(n)} : \Sigma^* \to \mathbb{R}$ is defined in Algorithm 3.

Let $\alpha = \sum_{i=p}^{\infty} 2^{-k\delta_i(s_2,s_1)}$. Then, as in the proof of Lemma 20, that $d' = \frac{1}{\alpha}\sum_{i=0}^{\infty} d^{(n)}$ is a constructive g-scaled s_2-gale.

Let $\mathtt{sa}'(w) = \sum_{i=0}^{|w|-1} \mathtt{sa}^{(n)'}(w)$. We define $\overline{p} : \Sigma^* \to [0, \infty]$ by

$$\overline{p}(w) = \begin{cases} \mathtt{sa}'(w)2^{-g(|w|,s_2)+|w|} & \text{if g is defined for } |w| \\ \mathtt{sa}'(w) & \text{otherwise.} \end{cases}$$

for strings w.

We define the function $p : \mathbf{C} \to [0, \infty]$ by

$$p(\omega) = \lim_{n\to\infty} \overline{p}(\omega[0 \ldots n-1]).$$

As in Lemma 20, if d is lower semicomputable, it is routine to verify that p is a measure of impossibility.

Also,

$$\frac{\int_{C(\omega[0\ldots n-1])} pdP}{2^{-g(n,s_2)}} \quad \geq \quad \mathtt{sa}'(\omega[0\ldots n-1]).$$

If d_1 succeeds on a binary sequence ω, then there is an n for which $d_2(\omega[0\ldots n-1]) > 2^{\delta_n(s_2,s_1)}$. Then, $\mathtt{sa}^{(n)'}(\omega[0\ldots n-1]) \geq M^{-1}2^{(1-k)\delta_n(s_2,s_1)}$. Hence,

$$\limsup_{n\to\infty} d_1(\omega[0\ldots n-1]) = \infty \Leftrightarrow \limsup_{n\to\infty} \mathtt{sa}'(\omega[0\ldots n-1]) = \infty.$$

If d_1 strongly succeeds on ω, then for all large enough lengths n, we have $\mathtt{sa}^{(n)'}(\omega[0\ldots n]) \geq M^{-1}2^{(1-k)\delta_n(s_2,s_1)}$, so

$$\liminf_{n\to\infty} d_1(\omega[0\ldots n-1]) = \infty \Leftrightarrow \liminf_{n\to\infty} \mathtt{sa}'(\omega[0\ldots n-1]) = \infty.$$

If, $g(n,s) = n$ for all n, then any g-scaled s-gale is a martingale. If there is a martingale that succeeds on ω, there is a measure of impossibility p to witness that ω is $s^{(g)}$-improbable with respect to P, and if there is a martingale that strongly succeeds on ω, then there is a measure of impossibility p to witness that ω is $s^{(g)}$-improbable.

If $g(n,s) > n$ for all large enough n, then the gale $d(w) = 2^{g(|w|,s)-g(|w|-1,s)}$ strongly succeeds on $\mathbf{C}$. Then, if we define $p(\omega) = \lim_{n\to\infty} d(\omega[0\ldots n-1])2^{-g(n,s)+g(n-1,s)-1}$, we can prove that p is a measure of impossibility such that $\mathbf{C}$ is strongly $s^{(g)}$-improbable.

Hence, if $s \in \mathcal{G}^{(g)}(X)$, we can conclude that

$$\limsup_{n\to\infty} \frac{\int_{C_{\omega[0\ldots n-1]}} p(\omega)d\omega}{2^{-g(n,s)}} = \infty,$$

and if $s \in \mathcal{G}^{(g)}_{str}(X)$, we can conclude that

$$\liminf_{n\to\infty} \frac{\int_{C_{\omega[0\ldots n-1]}} p(\omega)d\omega}{2^{-g(n,s)}} = \infty.$$

Conversely, let $s \in \mathcal{P}^{(g)}(X)$. Then there is a measure of impossibility p such that X is $s^{(g)}$-improbable with respect to p. Define a function $d : \Sigma^* \to \mathbb{R}$ as follows. For finite strings w,

$$d(w) = E[p|C_w]2^{g(|w|,s)-|w|}$$

It is routine to show that d is a lower semicomputable $s^{(g)}$-gale. Moreover, as in Lemma 20,

$$\limsup_{n\to\infty} \frac{\int_{C_{\omega[0\ldots n-1]}} p(x)dx}{2^{-g(n,s)}} = \limsup_{n\to\infty} d(\omega[0\ldots n-1]).$$

Thus, we can see that $s \in \mathcal{G}^{(g)}(X)$. Similarly, we can establish that if $s \in \mathcal{P}^{(g)}_{str}(X)$, then $s \in \mathcal{G}^{(g)}_{str}(X)$. □

Corollary 28. *(Alternate Characterization of Constructive Scaled Dimension) For any scale g, for any set $X \subseteq \mathbf{C}$, the constructive g-scaled dimension of X is* $\inf \mathcal{P}^{(g)}(X)$*, and the constructive g-scaled strong dimension of X is* $\inf \mathcal{P}^{(g)}_{str}(X)$*.*

Acknowledgments

The author wishes to thank Jack Lutz for posing the original question of characterization of constructive dimension, Jack Lutz and Xiaoyang Gu for very helpful discussions and very thoughtful suggestions, and Elvira Mayordomo for posing the question of whether a construction in a preliminary version of the paper extended to scaled dimension. The author is grateful to two anonymous reviewers who pointed out an error in the original proof of Lemmas 20 and 26, and suggested improvements in the presentation. The author would also like to thank two anonymous reviewers for suggestions.

References

[1] K. B. Athreya, J. M. Hitchcock, J. H. Lutz, and E. Mayordomo. Effective strong dimension, algorithmic information, and computational complexity. In *Proceedings of the 21st International Symposium on Theoretical Aspects of Computer Science*, pages 632–643. Springer-Verlag, 2004.

[2] R. Downey and D. Hirschfeldt. *Algorithmic randomness and complexity.* Book Draft, 2006.

[3] S. A. Fenner. Gales and supergales are equivalent for defining constructive hausdorff dimension. Technical Report cs.CC/0208044, Computing Research Repository, 2002.

[4] J. Hitchcock, J. Lutz, and E. Mayordomo. Scaled dimension and nonuniform complexity. *Journal of Computer and System Sciences*, 69:97–122, 2004.

[5] J. M. Hitchcock. Gales suffice for constructive dimension. *Information Processing Letters*, 86(1):9–12, 2003.

[6] J. M. Hitchcock, J . H. Lutz, and E. Mayordomo. The fractal geometry of complexity classes. *SIGACT News*, 36:24–38, 2005.

[7] A. N. Kolmogorov and V. A. Uspenksy. Algorithms and randomness. *Theory of Probability and Its Applications*, 32:389–412, 1987.

[8] P. Lévy. Propriétés asymptotiques des sommes de variables indépendantes ou enchainées. *Journal des mathématiques pures et appliquées. Series 9.*, 14(4):347–402, 1935.

[9] P. Lévy. *Théorie de l'Addition des Variables Aleatoires.* Gauthier-Villars, 1937 (second edition 1954).

[10] J. Lutz and E. Mayordomo. Dimensions of points in self-similar fractals. *submitted.*

[11] J. H. Lutz. Almost everywhere high non-uniform complexity. *Journal of Computer and System Sciences*, 44:220–258, 1992.

[12] J. H. Lutz. The quantitative structure of exponential time. In L. A. Hemaspaandra and A. L. Selman, editors, *Complexity Theory Retrospective II*, pages 225–254. Springer-Verlag, 1997.

[13] J. H. Lutz. Resource-bounded measure. In *Proceedings of the 13th IEEE Conference on Computational Complexity*, pages 236–248, New York, 1998. IEEE Computer Society Press.

[14] J. H. Lutz. Dimensions of individual strings and sequences. *Information and Computation*, 187(1):49–79, 2003.

[15] P. Martin-Löf. The definition of random sequences. *Information and Control*, 9(6):602–619, 1966.

[16] C. P. Schnorr. Klassifikation der Zufallsgesetze nach Komplexität und Ordnung. *Z. Wahrscheinlichkeitstheorie verw. Geb.*, 16:1–21, 1970.

[17] C. P. Schnorr. A unified approach to the definition of random sequences. *Mathematical Systems Theory*, 5:246–258, 1971.

[18] C. P. Schnorr. *Zufälligkeit und Wahrscheinlichkeit.* Springer-Verlag, Berlin, 1971.

[19] D. Sullivan. Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups. *Acta Mathematica*, 153:259–277, 1984.

[20] C. Tricot. Two definitions of fractional dimension. *Mathematical Proceedings of the Cambridge Philosophical Society*, 91:57–74, 1982.

[21] J. Ville. *Étude Critique de la Notion de Collectif.* Gauthier-Villars, Paris, 1939.

[22] V. G. Vovk and V. V. V'yugin. On the empirical validity of the bayesian method. *Journal of Royal Statistical Society B*, 55:253–266, 1993.

[23] V. V. V'yugin. Ergodic theorems for individual random sequences. *Theoretical Computer Science*, 207:343–361, 1998.