Expander Graphs

1 Expander graphs

In the preceding section, we considered regualar graphs. These graphs may not have high connectivity properties. For example, consider a "wrapped-around" grid of vertices in n rows and n columns, where a vertex at row i and column j is connected to vertices $((i+1) \mod n, j)$, $(i, (j+1) \mod n)$, $((i-1) \mod n, j)$ and $(i, (j-1) \mod n)$. Take the central square with $n/2$ rows and $n/2$ columns. This subset has $O(n^2)$ vertices. However, the cut that separates this subset from the rest of the graph has only $O(n)$ edges crossing it.

Expander graphs, in contrast, are sparse graphs with high connectivity properties. There are two ways of defining expanders, both of which are equivalent.

- Combinatorial expansion: A constant-degree regular graph G with n vertices is an *expander* if for every $S \subseteq G$ with $|S| < n/2$, a constant fraction of the edges incident on vertices in S are edges crossing from S to $G \setminus S$.
- Algebraic expansion: A constant-degree regular graph G is an *expander* if there is a constant $c > 0$ such that $\lambda(G) \leq 1 - c$.

Contrast the second definition with the bound from Lemma **??**. There, the bound would get closer to 1 as n increases, which is not ideal. However, in the above definition, the bound is independent of the size of the graph.

Note. By an expander graph, we usually mean a *family* of graphs, with one graph for each size.

1.1 The algebraic definition

Definition 1.1. If G is an n-vertex, d-regular undirected graph with $\lambda(G) \leq \lambda < 1$, then we say that G is an (n, d, λ)-graph.

A family of graphs $\langle G_n \rangle_{n \in \mathbb{N}}$ is an *expander graph family* if there are some constants $d \in \mathbb{N}$ and $\lambda < 1$ such that for every n, the graph G_n is an (n, d, λ) graph.

A main distinction between this and a family of regular graphs is that here, λ is uniformly bounded away from 1 for all n.

The smallest value for λ is $(1 - o(1))2\frac{\sqrt{d-1}}{d}$. This bound is called the *Alon-Boppana bound*. Graphs that meet this bound are called *Ramanujan graphs*.

1.2 Combinatorial expansion

Definition 1.2. An n-vertex d-regular graph $G = (V, E)$ is called an (n, d, ρ)-*combinatorial edge expander* if for every $S \subseteq V$ with $|S| < n/2$, we have

$$E(S, \overline{S}) \geq \rho d|S|, \tag{1}$$

where $\overline{S} = V \setminus S$, and for every subset of A, B of vertices, $E(A, B)$ is the set of edges $\{i, j\}$ with $i \in A$ and $j \in B$.

Using the probabilistic method, we can show that expanders exist. However, we will focus on an explicit construction of expanders later. We now establish the relation between combinatorial and algebraic definitions.

1.3 Algebraic expansion implies combinatorial expansion

Theorem 1.3. *If G is an (n, d, λ)-(algebraic) expander graph, then it is a $(n, d, (1-\lambda)/2)$ edge expander.*

This is a consequence of the following lemma.

Lemma 1.4. *Let $G = (V, E)$ be an (n, d, λ) graph, and $S \subseteq V$. Then*

$$E(S, \overline{S}) \geq (1-\lambda)\frac{d|S||\overline{S}|}{|S| + |\overline{S}|}. \tag{2}$$

Note that $|S| + |\overline{S}| = n$, but we write it in the above form for convenience in the following proof.

Proof. We define the vector $x \in \mathbb{R}^n$ where $x[i] = +|\overline{S}|$ if $i \in S$ and $x[i] = -|S|$ otherwise. Note that $\sum_{i=1}^{n} x[i] = |S||\overline{S}| - |\overline{S}||S| = 0$, so $x \perp \mathbf{1}$.

Then

$$\begin{aligned} \|x\|^2 &= \langle x, x \rangle \\ &= \sum_{i=1}^{n} x[i]x[i] \\ &= +|\overline{S}||\overline{S}||S| + |S||S||\overline{S}| \\ &= |S||\overline{S}|(|S| + |\overline{S}|). \end{aligned} \tag{3}$$

Let $A = ((a_{i,j}))$ be the $n \times n$ adjacency matrix.

Let $Z = \sum_{i,j} a_{i,j}(x[i] - x[j])^2$. Then

$$\begin{aligned} Z &= \sum_{i,j} a_{i,j}(x[i] - x[j])^2 \\ &= \sum_{i,j} a_{i,j}[x[i]^2 + x[j]^2 - 2x[i]x[j]]. \end{aligned}$$

Now, if $x[i] = x[j]$, *i.e.* i and j are in the same side of the cut, then the above becomes $2x[i]^2 - 2x[i]^2$. So such terms disappear. If $x[i] \neq x[j]$, then one of them is $|\overline{S}|$ and the other is $-|S|$. So the above term becomes $|\overline{S}|^2 + |S|^2 + 2|S||\overline{S}| = (|S| + |\overline{S}|)$. Thus we have

$$Z = \frac{2}{d}|E(S, \overline{S})|(|S| + |\overline{S}|)^2. \tag{4}$$

But, we also have

$$\begin{aligned} Z &= \sum_{i,j} a_{i,j}x[i]^2 - 2\sum_{i,j} a_{i,j}x[i]x[j] + \sum_{i,j} a_{i,j}x[j]^2 \\ &= 2\|x\|^2 - 2\langle x, Ax \rangle. \end{aligned} \tag{5}$$

We know that $x \perp \mathbf{1}$, hence $Ax \perp \mathbf{1}$. Further, $\|Ax\| \leq \lambda |x|$. Hence,

$$Z \geq 2(1-\lambda)\|x\|^2. \tag{6}$$

Thus, from (4), (5), and (3) we have

$$\frac{1}{d}|E(S,\overline{S})|(|S|+|\overline{S}|)^2 \geq (1-\lambda)\|x\|^2 = (1-\lambda)|S||\overline{S}|(|S|+|\overline{S}|),$$

yielding the result. □

Thus if a graph is algebraically an expander, then it is a combinatorial expander. A major advantage of the algebraic view is that we can show that the number of edges between any two disjoint subsets can be estimated easily:

Theorem 1.5 (Expander Mixing Lemma)**.** *Let $G = (V, E)$ be an (n, d, λ)-expander. Then for any $S, T \subseteq V$, we have*

$$\left| |E(S,T)| - \frac{d}{n}|S||T| \right| \leq \lambda d \sqrt{|S||T|}. \tag{7}$$

1.4 Combinatorial expansion implies algebraic expansion

Theorem 1.6. *If G is an (n, d, ρ) edge expander, then its second largest eigenvalue is at most $1 - \frac{\rho^2}{2}$. If G also has self-loops, then it is an $(n, d, 1-\varepsilon)$ expander where $\varepsilon = \min(2/d, \rho^2/2)$.*

Proof. Since λ is an eigenvalue, there is a $u \in \mathbb{R}^n$ such that $Au = \lambda u$. Since u is an eigenvector corresponding to λ, it also follows that $u \perp \mathbf{1}$. Write $u = u_+ + u_-$.

Assume that u_+ is non-zero on at most $n/2$ co-ordinates.

Let $Z = \sum_{i,j} a_{i,j} |u_+[i]^2 - u_-[j]^2|$.

First, we show that

$$Z \geq 2\rho \|u_+\|^2. \tag{8}$$

Then we show that

$$Z \leq \sqrt{8(1-\lambda)}\|u_+|^2. \tag{9}$$

□