CS 687 2013-14 Homework 1.

Due Date: August 31, 2013

Max Marks: 120

Instructions. Collaboration is encouraged, but copying is forbidden. After discussion, write the answer down by yourself.

Please mention all the people that you have collaborated with, for each answer.

Questions

1. Consider a fair die with 6 faces - the probability of each number appearing on any throw is equal to 1/6.

   What is the minimum number of times should a player throw the die before his/her probability of getting two consecutive sixes is greater than 1/2?

   A sequence of throws $T_0, T_1, \ldots$ is said to have two consecutive sixes if for some natural number i, both T_i and T_{i+1} are sixes. For example, $1, 6, 6$ and $1, 2, 6, 6$ both have two consecutive sixes.

   (Scheme: Half the points for a numerical solution or a code. Full points for an analytical, closed form expression.) [10 points]

   Extra Credit: Suppose the probability of each n, $1 \leq n \leq 6$, is given by the probability vector $(p_1, p_2, \ldots, p_6)$. How many times should a layer throw the die before the probability of getting two consecutive sixes is greater than 1/2?)

2. Let A and B be two finite sets. Prove that if $f : A \to B$ is one-to-one, then $|f(A \cup B)| = |f(A)| + |f(B)|$.

   Prove that for a function $f : A \to B$, $|f(A \cup B)| \leq |f(A)| + |f(B)|$. [10 points]

3. Let $S = \{0^{2^i} \mid i \in \mathbf{N}\}$. Is its power set $\mathcal{P}(\mathcal{S})$ countable? Justify your answer. [10 points]

4. Let $\mathbf{R}$ represent the set of reals, and $\mathbf{N}$ represent the set of natural numbers. Then $\mathbf{R}^{\mathbf{N}}$ is the set of infinite sequences of reals.

   Prove or disprove: $\mathbf{R}^{\mathbf{N}}$ has the same cardinality as $\mathbf{R}$. [15 points]

5. Prove that if languages A and B are decidable, then so are $A \cup B$, $A \cap B$ and $A \setminus B$. [10 points]

6. A language L is defined to be *decidable* if L and L^c are acceptable. Prove that L is decidable if and only if there is a Turing Machine M such that M accepts every string in L and rejects every string not in L^c. (It halts on all inputs.) [10 points]

7. Prove that if A and B are computably enumerable, then so are $A \cup B$ and $A \cap B$. Show an example where A and B are computably enumerable, but not $A \setminus B$. [10 points]

8. Prove that for all computably enumerable languages A and B, we can computably define disjoint computably enumerable languages $A' \subseteq A$ and $B' \subseteq B$ such that $A' \cup B' = A \cup B$.

   (That is to say, given two Turing Machines M_A accepting A and M_B accepting B, it is possible for a Turing Machine to enumerate A' and B'.) [15 points]

9. Prove that every infinite computably enumerable language has an infinite decidable subset. [15 points]

10. Let $f_0, f_1, \ldots$ be an enumeration (not necessarily computable) of total computable functions from natural numbers to natural numbers.

    Prove that there cannot be a universal total computable function $\phi : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ such that for any number e, we have $\phi(e, n) = f_e(n)$. [15 points]