Arithmetic Progressions and Symbolic Dynamical Systems

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van der Waerden’s Theorem

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Theorem: Van der Waerden 1927

Suppose $\mathbb{N}$ is partitioned into two sets S_1 and S_2. Then either S_1 or S_2 has arbitrarily long arithmetic progressions — i.e. $\exists S_i$ such that for every $k \geq 2$, there are integers a and b such that have

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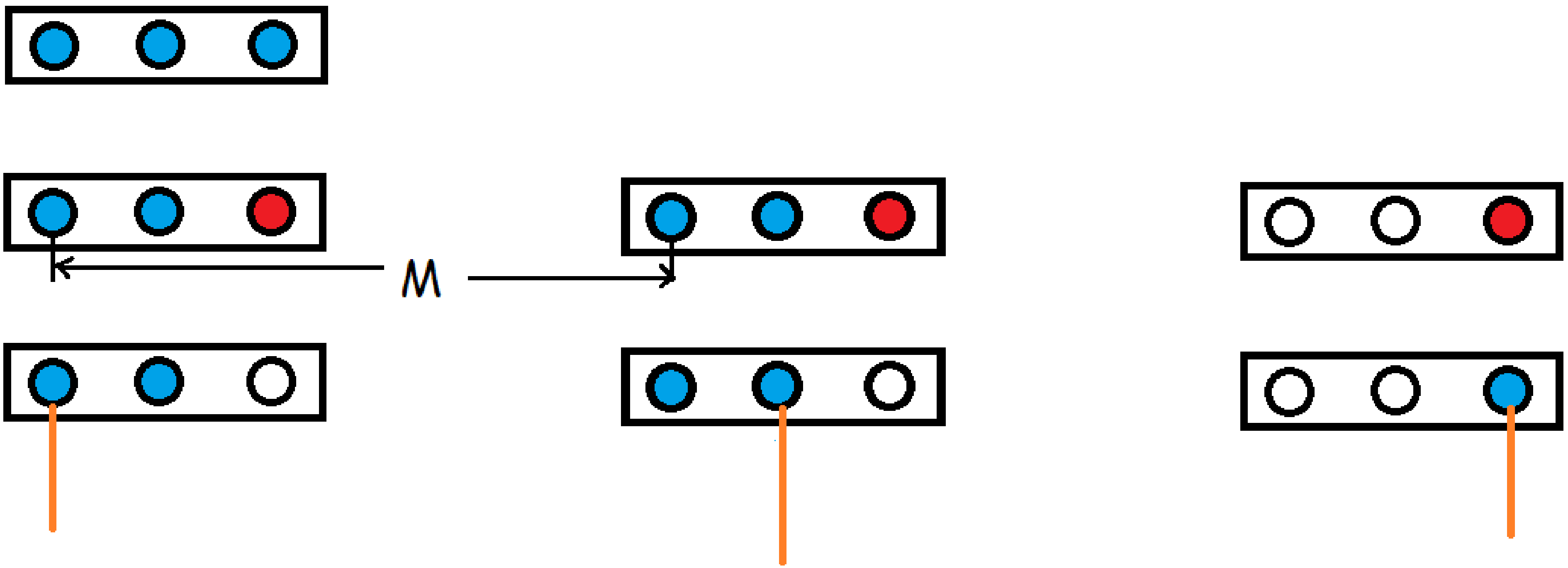
Divide 325 into 65 blocks, [1-5], [6-10], $\ldots$, [321-325]. Each number is colored either red or blue (say).

There are 32 possible block colorings. Pigeonhole $\implies$ 2 blocks in the first 33 are colored the same.

Proof of vdW



Erdős' Conjecture

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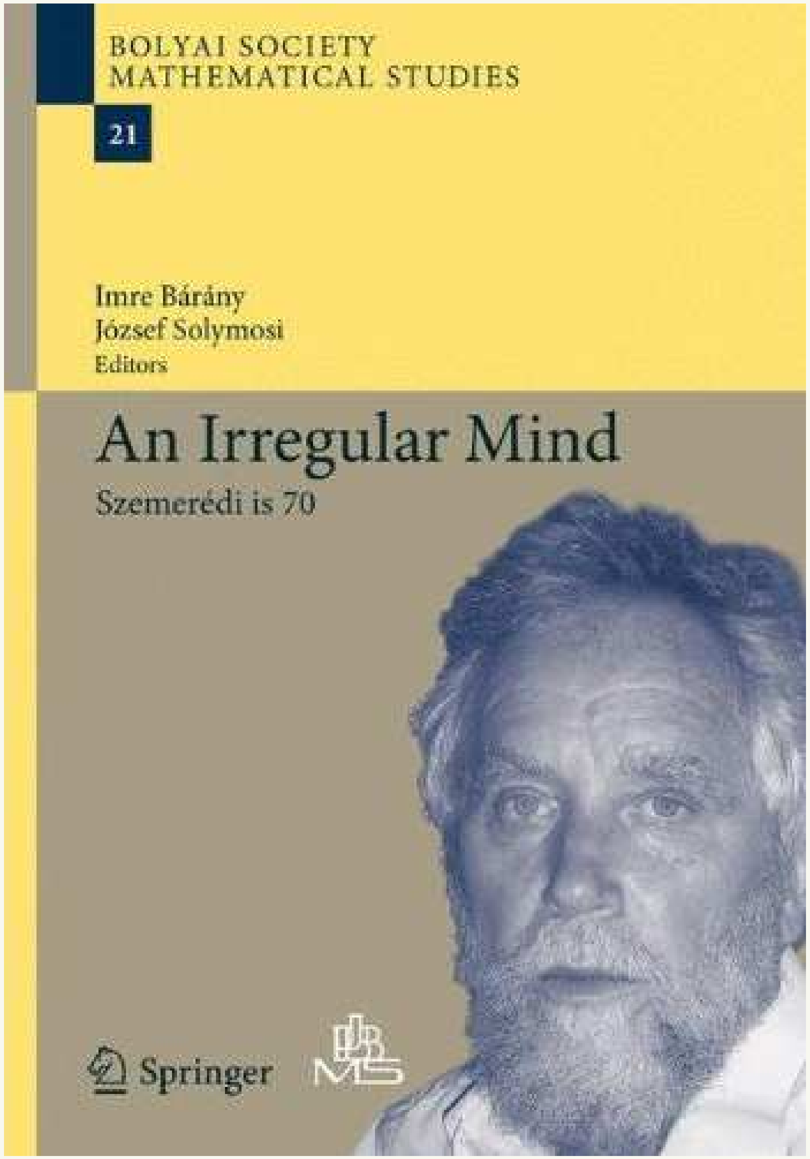
Theorem: (Szemerédi 1975)

Erdős conjecture holds.

Proof uses his "regularity lemma".


BOLYAI SOCIETY
MATHEMATICAL STUDIES
21
Imre Bárány
József Solymosi
Editors
An Irregular Mind
Szemerédi is 70
Springer

Some highlights

1. Roth 1956 Erdős Conjecture holds for length 3 A.P.
2. Szemerédi’s Theorem 1975
3. Furstenberg’s ergodic theory proof 1978
4. Gowers’ Fourier Analytic proof, 1996
5. Green-Tao A.P. in primes

Infinitude of Primes — A topological proof

A “Proof from the Book”

Theorem

There are infinitely many prime numbers.

The following proof is by Hillel Furstenberg, 1955.

Proof. Consider the topology on $\mathbb{Z}$ where U is open if and only if it is empty, or a union of arithmetic progressions of the form

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$\{-1, +1\}^c = \cup_{p \text{ prime}} A(p, 0)$.

$\{-1, 1\}^c$ cannot be closed. Each $A(p, 0)$ is closed. If the number of primes were finite, then the RHS would be closed! □

Topological Dynamics

Further connections between topological dynamics and integer sets.

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Pigeonhole principle and Recurrence in Open Covers

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Let (X, T) be a toplogical dynamical system, and $(U_\alpha)_{\alpha \in \Omega}$ be an open cover of X. Then there is a U_α in the cover for which for infinitely many n, $U_\alpha \cap T^n U_\alpha \neq \emptyset$.

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Consider $O = \{n \in \mathbb{Z} \mid T^n x \in U_i\}$. Pick some $n_0 \in O$.

$\forall n \in O$, $\quad T^{n_0}x = T^{n_0 - n}T^n x$. Hence $T^{n_0}x \ \in \ U_i \cap T^{n_0-n}U_i$.

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Hence for infinitely many n, $U_i \cap T^{n_0 - n}U_i \neq \emptyset$.

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By recurrence in open covers,

$$\exists c \in \Omega \;\; \exists^{\infty} n \;\; U_c \cap T^n U_c \neq \emptyset. \tag{1}$$

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Since X_a is the orbit closure of a, there is a $k \in \mathbb{Z}$ such that $T^k a \in U_c \cap T^n U_c$. That is, $a_{-k} = c$ and $a_{-k+n} = c$. This is true for all n in (1).

van der Waerden’s Theorem and Multiple Recurrence in Open Covers

The version of recurrence in tds which is equivalent to Van der Waerden's theorem is the following.

Theorem: Multiple Recurrence in Open Covers

Let (X,T) be a topological dynamical system and $(U_\alpha)_{\alpha\in\Omega}$ be an open cover of X. Then there is a U_α in the cover such that

$$\forall k \geq 2 \;\; \exists n > 0 \qquad U_\alpha \cap T^n U_\alpha \cap \cdots \cap T^{(k-1)n} U_\alpha \neq \emptyset.$$

Szemerédi's Theorem

Dynamical Systems view of Szemerédi's Theorem

For Szemerédi's theorem, we now have to consider *measure* as well.

Definition

A *measure-preserving topological dynamical system* is a quadruple $(X, \mathcal{X}, \mu, T)$ is a space where

- X is a compact topological space,
- $\mathcal{X}$ is a σ-algebra on X,
- μ a probability measure on $\mathcal{X}$ and
- $T : X \to X$ is a measure-preserving homeomorphism.

Fursteneberg Multiple Recurrence Theorem

Multiple Recurrence

Theorem: Multiple Recurrence Theorem

Let $(X, \mathcal{X}, \mu, T)$ be a mtds. Then for any $E \in \mathcal{X}$ with $\mu(E) > 0$, we have

$$\mu(E \cap T^n E \cap \cdots \cap T^{(k-1)n} E) > 0.$$

Furstenberg Correspondence Principle

Lemma

Let $(X, \mathcal{X}, \mu, T)$ be as in the FMRT, and E have positive measure. Then there is an F, $\mu(F) > 0$ such that for every x in F,

$$\{n \in \mathbb{Z} \mid T^n x \in E\}$$

has positive upper density.

Proof of Lemma

Proof.

- Define $\delta_N(x)$ to be the frequency with which $T^{-N}x, \ldots, T^N x$ visits E. Then the expected value of δ_N is $\mu(E)$.
- By an Egorov-style argument, show that the probability of

$$\left\{ x \in X \mid \delta_N(x) \geq \frac{1}{2}\mu(E) \right\}$$

  is at least $1/2\mu(E)$.
- Then F is the set $\bigcap_N \bigcup_{m>N} A_m$.

□

Szemerédi's Theorem $\Rightarrow$ Furstenberg Multiple Recurrence Theorem

Let AP_k denote the set of k-length arithmetic progressions in $\mathbb{Z}$. For each $\alpha = (a_1, \ldots, a_k)$ which is an A.P., define

$$B_\alpha = \{x \in X \mid T^{a_1}x, \ldots, T^{a_k}x \in E\}.$$

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Thus some $b, n \in \mathbb{Z}$ exist such that $T^b B_a \subseteq E \cap T^n E \cap \cdots \cap T^{(k-1)n} E$, and $\mu(T^b B_a) > 0$. [A.P. starting with 0]

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Let A have positive upper density, represented as $a \in 2^{\mathbb{Z}}$. Consider the system $(2^{Z}, T)$, where $TB = B + 1$ is the right shift.

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If we can find an invariant measure μ with $\mu(E) > 0$, then by FMRT, we can conclude Szemerédi's theorem.

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If we can find an invariant measure μ with $\mu(E) > 0$, then by FMRT, we can conclude Szemerédi's theorem.

The invariant measure is constructed using the Banach-Alaoglu theorem.

Furstenberg Multiple Recurrence - the “random” case

Weak Mixing Systems - Bernoulli Systems

Consider $B = (2^{\mathbb{Z}}, \mathcal{B}, T, \mu)$ where T is the right-shift and μ the product measure specified by $\mu(0)$ and $\mu(1)$.

Theorem. *B has multiple recurrence.*

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Hence $\mu(E \cap T^n E \cap \cdots \cap T^{(k-1)n} E)$ = $\mu(E)^k$ by independence and the measure preservation of T. $\mu(E)^k > 0$. $\square$

Furstenberg Multiple Recurrence - the “structured” case

Kronecker Systems - Irrational Rotation

Consider $X = (\mathbb{R}/\mathbb{Z}, T_\alpha)$ where $T_\alpha(x) = (x + \alpha) \mod 1$. This is an *almost-periodic* system —

$$\forall \varepsilon > 0 \quad \forall x \in \mathbb{R}/\mathbb{Z} \quad \exists N \forall n \qquad ||T_\alpha^n x - T_\alpha^{n+N} x|| < \varepsilon.$$

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For any n, $|(x + n\alpha) \mod 1 - (x + (n + Q)\alpha) \mod 1| < \varepsilon/k$.

i.e. $\forall 0 \leq j \leq k - 1$, $|x - T_\alpha^{jQ} x| \leq j\varepsilon/k \leq \varepsilon$. Hence $T^{jQ} x \in V$. $\square$

Further...

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Lift multiple recurrence from the bottommost system all the way to X.

Effective Versions