Randomness and effective dimension of continued fractions

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Abstract

Recently, Scheerer [20] and Vandehey [22] showed that normality for continued fraction expansions and base-b expansions are incomparable notions. This shows that at some level, randomness for continued fractions and binary expansion are different statistical concepts. In contrast, we show that the continued fraction expansion of a real is computably random if and only if its binary expansion is computably random.

To quantify the degree to which a continued fraction fails to be effectively random, we define the effective Hausdorff dimension of individual continued fractions, explicitly constructing continued fractions with dimension 0 and 1.

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1 Introduction

Kolmogorov initiated the program of proving that all practical applications of randomness are consequences of incompressibility [10]. A landmark achievement in the theory of computation realizing Kolmogorov's program is Martin-Löf's definition of an individual random binary sequence using constructive measure [16]. Alternative, equivalent characterizations using martingales [21] and incompressible sequences [11], [6], [1], establish that the definition of an individual random binary sequence is mathematically robust. This has led to a deep and rich theory interacting fruitfully with computability theory, probability theory and dynamical systems (see for example, [12], [3], [19]).

In this work, we study the concept of an individual random continued fraction. An important question is whether randomness of a real is preserved when translating from one representation to another, for example, from base 2 expansion to base 3 expansion, or from binary expansion to continued fraction expansion. Recent elegant constructions by Vandehey and Scheerer show that continued fraction normals and normals in base-b are incomparable sets [22], [20]. In contrast, Nandakumar [18] remarks that the binary expansion of a real is Martin-Löf random if and only if its continued fraction is. We extend this result using martingales, and show that the continued fraction of a real is *computably* random if and only if its binary expansion is.

To quantify the degree of non-randomness, the topological notion of Hausdorff dimension [8] has been effectivized in computability and complexity theory in a series of works by Lutz [14], Lutz and Mayordomo [15], Mayordomo [17], Fernau and Staiger [5], and others. Generalizing the definition of random continued fractions using martingales, we define the effective Hausdorff dimension of sets of continued fractions, and of individual continued fractions, in the spirit of Lutz [14]. We construct examples of continued fractions with dimensions 0 and 1.

The tools and techniques for base-2 randomness do not lend themselves easily to continued fractions, which we can view as infinite sequences over a countably infinite alphabet. Topologically, this is a non-compact space. Further, the canonical shift-invariant measure on the space of continued fractions in $[0,1]$ is the Gauss measure, which is not a product measure, or even a Markov distribution [4], [2]. A study of effective Hausdorff dimension in this setting is new.

Our main contributions are - martingale-based definitions of Martin-Löf random and computable random continued fractions, showing the preservation of Martin-Löf randomness and computable randomness when converting from binary expansion to continued fractions and vice versa, and a basic statistical property of random sequences. Further, we define effective Hausdorff dimension of sets of continued fractions and individual continued fractions using s-gales, and give explicit constructions of continued fractions with dimensions 0 and 1. We develop techniques and approximation methods related to Gauss measure, which may be of independent interest.

2 Preliminaries

Let $\mathbb{N}$ be the set of positive natural numbers, $\mathbb{N}^*$ be the set of finite sequences of natural numbers, and $\mathbb{N}^\infty$ be the set of infinite sequences of natural numbers. If a finite sequence $v \in \mathbb{N}^*$ is a prefix of another finite sequence $w \in \mathbb{N}^*$ or an infinite sequence $X \in \mathbb{N}^\infty$, we represent it respectively by $v \sqsubseteq w$ and $v \sqsubseteq X$. If $v, w \in \mathbb{N}^*$, their concatenation is written as vw. λ denotes the empty string.

We identify any finite string $(a_1, \dots, a_n) \in \mathbb{N}^*$, and any infinite sequence $\langle a_i \rangle_{i \in \mathbb{N}}$ with

$$0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}} \quad \text{and} \quad 0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots}} \tag{1}$$

respectively. We denote this respectively as the finite continued fraction $[0; a_1, \dots, a_n]$ and the infinite continued fraction $[0; a_1, \dots]$. The *continued fraction cylinder* $C_{[0;a_1,\dots,a_k]}$ is the set of infinite continued fractions with $[0; a_1, \dots, a_k]$ as a prefix.

If $v \in \mathbb{N}^*$, then the number of integers in v is denoted $|v|$. For $j \in \mathbb{N}$, $v \upharpoonright j$ denotes the substring consisting of the first j integers in v when $j \leq |v|$, and v itself, otherwise. For $X \in \mathbb{N}^\infty$ and $j \in \mathbb{N}$, $X \upharpoonright j$ denotes the substring consisting of the first j integers in X.

In this work, we consider the probability space $(\mathbb{N}^\infty, \mathcal{B}(\mathbb{N}^\infty), \gamma)$ where $\mathcal{B}(\mathbb{N}^\infty)$ is the Borel σ-algebra generated by the cylinders and γ is the Gauss measure defined on any $A \in \mathcal{B}(\mathbb{N}^\infty)$ by $\gamma(A) = \frac{1}{\log 2} \int_A \frac{1}{x+1}\, dx$. The Gauss measure is a translation-invariant probability on the space of continued fractions [4], [2].

Similar notations apply for the binary expansions of reals. We designate the binary alphabet $\{0, 1\}$ by Σ. Analogous with the notation for integers, let Σ^* denote the set of finite binary strings, and Σ^∞ the set of infinite binary sequences. We use λ for the empty string. For any $w \in \Sigma^*$, the *binary cylinder* C_w is the set of all infinite binary sequences with w as a prefix. The probability space on binary sequences is $(\Sigma^\infty, \mathcal{B}(\Sigma^\infty), \mu)$ where $\mathcal{B}(\Sigma^\infty)$ is the Borel σ-algebra on Σ^∞, and μ is the Lebesgue (uniform) probability measure defined for every Borel set A by $\mu(A) = \int_A x dx$.

For $w \in \Sigma^*$, we denote $\mu(C_w)$ by $\mu(w)$, and analogously for $v \in \mathbb{N}^*$ and γ.

3 Useful estimates for continued fractions and the Gauss measure

For $[0; v_1, \ldots, v_n]$, denote the rational represented by $v \upharpoonright k$ by $\frac{p_k}{q_k}$. This is called the k^{th} *convergent* of v. The standard continued fraction recurrence for computing convergents is given by (see for example, Khinchin [9])

$$p_{-1} = 1, \quad p_1 = 0, \quad p_n = v_n p_{n-1} + p_{n-2},$$
$$q_{-1} = 0, \quad q_1 = 1, \quad q_n = v_n q_{n-1} + q_{n-2}.$$

It follows that $\mu([0; v_1, \ldots, v_k]) = \frac{1}{q_k(q_k+q_{k-1})}$ for all $2 \leq k \leq n$.

▶ **Lemma 1.** *Let $C_{[0;a_1,\ldots,a_k]}$ be the cylinder set of an arbitrary finite continued fraction and $C'_{b_1\ldots b_k}$ be the cylinder set of an arbitrary binary string of length k. Then, $\mu(C_{a_1,a_2\ldots a_n}) \leq \mu(C'_{b_1,b_2\ldots b_n})$.*

The following estimate, which we can easily establish, shows a fairly tight relationship between Lebesgue measure and Gauss measure. The proof uses the fact that the Radon-Nikodym derivative $\frac{d\gamma}{d\mu} = \frac{1}{1+x}$ is bounded in $[0, 1]$.

▶ **Lemma 2.** *For any subinterval B of the unit interval, we have $\frac{1}{2\ln 2}\mu(B) \leq \gamma(B) \leq \frac{1}{\ln 2}\mu(B)$.*

4 Martingales on Continued fraction expansions

The notion of binary supermartingales and their success sets is well-known in the study of algorithmic randomness and resource-bounded measure [12], [19], [3]. We recall the binary notion, and then define the notion of continued fraction supermartingales, by replacing the measure appropriately.

▶ **Definition 3.** *[3] A* binary martingale *$d : \Sigma^* \to [0, \infty)$ is a function with $d(\lambda) < \infty$ and such that for every $v \in \Sigma^*$, $d(v) = \frac{d(v0)+d(v1)}{2}$. We say that $d : \Sigma^* \to [0, \infty)$ is a* binary supermartingale *if $d(\lambda) < \infty$, and the equality above is replaced with a $\geq$.*

A supermartingale or a martingale d succeeds *on $X \in \Sigma^\infty$, denoted $X \in S^\infty[d]$, if $\limsup_{n\to\infty} d(X \upharpoonright n) = \infty$, and* strongly succeeds *on X, denoted $X \in S^\infty_{str}[d]$, if $\liminf_{n\to\infty} d(X \upharpoonright n) = \infty$.*

Analogously, we define the following.

▶ **Definition 4.** *A* continued fraction martingale *$d : \mathbb{N}^* \to [0, \infty)$ is a function with $d(\lambda) < \infty$ and such that for every $v \in \mathbb{N}^*$, $d(v)\gamma(C_v) = \sum_{n\in\mathbb{N}} d(vn)\gamma(C_{vn})$. We say that $d : \mathbb{N}^* \to [0, \infty)$ is a* continued fraction supermartingale *if $d(\lambda) < \infty$, and the equality above is replaced with a $\geq$.*

A supermartingale or a martingale d succeeds *on an infinite sequence X, denoted $X \in S^\infty[d]$, if $\limsup_{n\to\infty} d(X \upharpoonright n) = \infty$, and* strongly succeeds *on X, denoted $X \in S^\infty_{str}[d]$, if $\liminf_{n\to\infty} d(X \upharpoonright n) = \infty$.*

We view the value $d(w)$ as the capital that the martingale has if the outcome is w. Thus, a martingale is a "fair" betting condition on continued fractions where the expected value (with respect to the Gauss measure) of the capital after a bet is equal to the expected value before the bet. The reason for selecting Gauss measure in particular as the "canonical" distribution is that it is translation invariant with respect to the continued fraction expansion, which is necessary to study statistical properties of sequences like normality.

The following is a consequence of the definition of martingales.

▶ **Lemma 5.** *Let $d : \mathbb{N}^* \to [0,\infty)$ be a supermartingale. Let $v \in \mathbb{N}^*$ and $S \subseteq \mathbb{N}^*$ be a prefix-free set where every $w \in S$ is an extension of v with $|w| \leq k$, $k \in \mathbb{N}$. Then $\sum_{w \in S} d(w)\gamma(w) \leq d(v)\gamma(v)$.*

Now, we impose computability restrictions on the (super)martingale functions, analogous to the existing notions for the computability of martingales on finite alphabets [3].

▶ **Definition 6.** *A function $d : \mathbb{N}^* \longrightarrow [0,\infty)$ is called* computably enumerable *(alternatively,* lower semicomputable*) if there exists a total computable function $\hat{d} : \mathbb{N}^* \times \mathbb{N} \longrightarrow \mathbb{Q} \cap [0,\infty)$ such that the following two conditions hold.*

- ***Monotonicity** : For all $w \in \mathbb{N}^*$ and for all $n \in \mathbb{N}$, we have $\hat{d}(w,n) \leq \hat{d}(w,n+1) \leq d(w)$.*
- ***Convergence** : For all $w \in \mathbb{N}^*$, $\lim_{n\to\infty} \hat{d}(w,n) = d(w)$.*

A real number r is said to be lower semicomputable if there is a total computable function $\hat{r} : \mathbb{N} \to \mathbb{Q}$ such that for every $n \in \mathbb{N}$, $\hat{r}(n) \leq \hat{r}(n+1) \leq r$, and $\lim_{n\to\infty} \hat{r}(n) = r$. Note that if d is a lower semicomputable supermartingale, then for every $v \in \mathbb{N}^*$, $d(v)$ is a lowersemicomputable real, uniformly in $\mathbb{N}$.

▶ **Definition 7.** *A function $d : \mathbb{N}^* \to [0,\infty)$ is called* computable *if there is a total computable function $\hat{d} : \mathbb{N}^* \times \mathbb{N} \to \mathbb{Q} \cap [0,\infty)$ such that for every $w \in \mathbb{N}^*$ and $n \in \mathbb{N}$, we have $|\hat{d}(w,n) - d(w)| \leq 2^{-n}$.*

Note. By replacing $\mathbb{N}^*$ with Σ^*, we get the analogous computability notions for binary supermartingales. For a computable function d, it is sufficient for the witness $\hat{d}$ that for some $f : \mathbb{N} \to [0,\infty)$, where f is a monotone computable function decreasing to 0 as $n \to \infty$, $|\hat{d}(w,n) - d(w)| \leq f(n)$.

For c.e. sequences of lower semicomputable martingales, we have the following universality result.

▶ **Theorem 8.** *If $\{d_1, d_2, \dots\} : \mathbb{N}^* \to [0,\infty)$ is a computably enumerable sequence of lower semicomputable martingales then there exists a lower semicomputable martingale d that succeeds on $\cup_{i=1}^{\infty} S^{\infty}[d_i]$, and which strongly succeeds on $\cup_{i=1}^{\infty} S^{\infty}_{str}[d_i]$.*

We now define individual random continued fractions for the above computability notions. Random sequences are those on which martingales fail to make unbounded amounts of money.

▶ **Definition 9.** *We call a continued fraction $X \in \mathbb{N}^\infty$* Martin-Löf random *if no lower semicomputable supermartingale succeeds on X and* computably random *if no computable supermartingale succeeds on X.*

As it is well-known in the binary case using the "savings account trick" (see for example, [3] or [19]), the following theorem states that the notion of success and strong success coincide when we study Martin-Löf and computable randomness.

▶ **Theorem 10.** *If $d : \mathbb{N}^* \to [0,\infty)$ is a supermartingale which succeeds on $X \in \mathbb{N}^*$, then there is a supermartingale $g : \mathbb{N}^* \to [0,\infty)$ and such that $\lim_{n\to\infty} g(X \upharpoonright n) = \infty$. Moreover, if d is lower semicomputable, then so is g. If d is computable, then there is a function $s : \mathbb{N}^* \to [0,\infty)$ which is monotone over lengths of inputs, such that $g \geq s$ and $\lim_{n\to\infty} s(X \upharpoonright n) = \infty$, where g nand s are computable functions.*[1]

[1] s is called the "savings account" of g.

We can show that basic stochastic properties are satisfied by continued fraction randoms.

▶ **Theorem 11.** *Suppose $X \in \mathbb{N}^\infty$ is computably random. Then every positive integer appears infinitely often in X.*

5 Continued fraction non-randoms are binary non-random

The following lemmas are crucial in converting betting strategies on binary expansions into those on continued fractions, and conversely.

▶ **Lemma 12.** *Let $0 \le a < b \le 1$, and $\left[\frac{m}{2^k}, \frac{m+1}{2^k}\right)$, where $0 \le m < 2^k$, be the smallest dyadic interval that covers $[a, b)$. Then $\frac{1}{2^k} \le 4(b-a)$.*

▶ **Lemma 13.** *Let $0 \le a < b \le 1$, and $\left[\frac{m}{2^k}, \frac{m+1}{2^k}\right)$, where $0 \le m < 2^k$, be the largest dyadic interval which is a subset of $[a, b)$. Then $\frac{1}{2^k} \ge \frac{1}{4}(b-a)$.*

Now we show that if there is a martingale which succeeds on the continued fraction on a real number x, then there is a martingale that succeeds on its binary expansion with similar computability properties.

▶ **Theorem 14.** *Let $x \in (0, 1)$ be an irrational with continued fraction expansion X and binary expansion B. Then the following hold.*

1. *If X is non-Martin-Löf random, then its B is non Martin-Löf random.*
2. *If X is not computably random, then B is not computably random.*

Proof. Let X and B be as given.

Let $d : \mathbb{N}^* \to [0, \infty)$ be a c.e. supermartingale which succeeds on X. By Theorem 10, we can assume that $\liminf_{n\to\infty} d(X \upharpoonright n) = \infty$, equivalently, for every integer M, for all sufficiently large prefix lengths n, $d(X \upharpoonright n) \ge M$. We construct a c.e. martingale $h : \Sigma^* \to [0, \infty)$ which succeeds on B, using the martingale d.

Note that for an arbitrary $w \in \{0, 1\}^*$, the continued fraction cylinder enclosing C_w may not coincide exactly with C_w, and that certain intervals may overlap with both C_{w0} and C_{w1}. First, we introduce some notation to define the martingale.

Let $w \in \Sigma^*$ and $v \in \mathbb{N}^*$ be the continued fraction such that C_v is the smallest cylinder enclosing C_w. We classify the extensions of v as follows. Let $I(w) = \{vi \mid i \in \mathbb{N}, C_{vi} \subseteq C_w\}$ be the set of cylinders which are contained in C_w. Let $P(w) = \{vi \mid i \in \mathbb{N}, C_{vi} \cap C_w \neq \emptyset, C_{vi} \not\subseteq C_w\}$ be the set of cylinders which partially intersect C_v, but are not contained in it. Then, let

$$h(w) = \sum_{y \in I(w)} d(y) \frac{\gamma(y)}{\mu(w)} \frac{1}{2} \sum_{y \in P(w)} d(y) \frac{\gamma(y)}{\mu(w)}. \tag{2}$$

Since $\mu(w0) = \mu(w1) = \frac{\mu(w)}{2}$, we have that

$$[h(w0) + h(w1)] \frac{\mu(w)}{2} = \sum_{y \in I(w0) \cup I(w1)} d(y)\gamma(y) + \sum_{y \in P(w0) \cap P(w1)} d(y)\gamma(y) + \frac{1}{2} \sum_{y \in P(w0) \oplus P(w1)} d(y)\gamma(y),$$

where $\oplus$ denotes the symmetric difference of sets. Note that every $y \in I(w0) \cup I(w1) \cup (P(w0) \cap P(w1))$ is an extension of some $v \in I(w)$. By Lemma 5, we have

$$\sum_{y \in I(w0) \cup I(w1) \cup (P(w0) \cap P(w1))} d(y)\gamma(y) \leq \sum_{v \in I(w)} d(v)\gamma(v).$$

Further, every $y \in P(w0) \oplus P(w1)$ is an extension of some $v \in P(w)$. Hence

$$\sum_{y \in P(w0) \oplus P(w1)} d(y)\gamma(y) \leq \sum_{v \in P(w)} d(v)\gamma(v).$$

We have

$$[h(w0) + h(w1)]\frac{\mu(w)}{2} \leq \left(\sum_{v \in I(w)} d(v)\gamma(v) + \frac{1}{2} \sum_{v \in P(w)} d(v)\gamma(v) \right) = h(w)\mu(w),$$

whence h is a supermartingale.

Let M be an arbitrary positive real, and let $v \sqsubseteq X$ be a prefix such that for all longer prefixes, $d(v) \geq M$.

Let $w \sqsubseteq B$ be the string designating the largest binary cylinder $C_w \subseteq C_v$. We show that $h(w) \geq cM$ for some constant $c > 0$ which is independent of w, v, and M.

By Lemma 13, we know that the largest dyadic interval which is a subset of C_v has Lebesgue measure at least $1/4$ of the Lebesgue measure of C_v. Thus,

$$\gamma(C_v \cap C_w) \geq \frac{\mu(C_v \cap C_w)}{2\ln(2)} \geq \frac{\mu(C_v)}{8\ln(2)} \geq \frac{\gamma(C_v)}{8}.$$

The first and third inequalities above are consequences of Lemma 2 (see also [4], Section 3.2) and the second, Lemma 13.

By definition, we have

$$\begin{aligned} h(w) &\geq M \left[\sum_{y \in I(w)} \gamma(y) 2^{|w|} + \frac{1}{2} \sum_{i \in P(w)} \gamma(y) 2^{|w|} \right] \\ &\geq \frac{M}{2} \left[\sum_{y \in I(w)} \gamma(y) 2^{|w|} + \sum_{y \in P(w)} \gamma(y) 2^{|w|} \right] \\ &\geq \frac{M}{2} \gamma(C_v \cap C_w) 2^{|w|}. \end{aligned}$$

From the bound above, we obtain

$$h(w) \geq \frac{M}{2} \frac{\gamma(C_v \cap C_w)}{\mu(C_w)} \geq \frac{M}{16} \frac{\gamma(C_v)}{\mu(C_w)} = \frac{M}{32 \ln 2} \frac{\mu(C_v)}{\mu(C_w)} \geq \frac{M}{32 \ln(2)},$$

where the last inequality follows from the fact that $C_v \supseteq C_w$. Thus h succeeds on the same real.

If d is lower semicomputable, from equation (2), it is clear that h is the sum of lower semicomputable terms involving a computable decision (i.e. $i \in I(wb)$ and $i \in P(wb)$). Hence h is a lower semicomputable function.

Now, suppose d is computable. Observe crucially that $|I(wb)| < \infty$ for one bit $b \in \{0, 1\}$. Assume, without loss of generality, that $|I(w0)| < \infty$. Hence, $h(w0)$ is a sum of finitely many computable terms, involving a computable decision. Moreover, $h(w1) = \frac{h(w)-h(w0)}{2}$ is a difference of computable terms. It follows that h is computable. ◀

6 Binary non-randoms are continued fraction non-random

We now show that if the binary expansion of a real number is non-Martin-Löf-random, then so is its continued fraction expansion.

▶ **Theorem 15.** *Let x be an irrational in $[0,1]$ with continued fraction expansion X and binary expansion B. If B is not Martin-Löf random, then X is not a Martin-Löf random continued fraction. If B is not computably random, then X is not a computably random continued fraction.*

Proof. Let $d : \Sigma^* \to [0,\infty)$ be a martingale with $B \in S^{\infty}_{\text{str}}[d]$. By Lemma 25, we may assume that $d \geq 2^{-c}$ for some $c \in \mathbb{N}$, $c > 0$.

Construct a collection of sets $\langle \mathcal{L}_v \rangle_{v \in \mathbb{N}^*}$ by letting $\mathcal{L}_\lambda = \{\lambda\}$ and

$$\mathcal{L}_{vi} = \{w \in \Sigma^* \mid (\exists u \sqsubseteq w)\ u \in \mathcal{L}_v,\ (\nexists u \sqsubset w)\ u \in \mathcal{L}_{vi},\ C_w \subseteq C_{vi}\}. \tag{3}$$

Dyadic rationals are dense in $[0,1]$. Hence $\mathcal{L}_v$ contains a unique prefix of every irrational in C_{vi}. By construction, every $\mathcal{L}_v$ is a prefix-free set. Further, membership of w in $\mathcal{L}_v$ can be decided by ensuring that for every prefix $v' \sqsubset v$, there is some $u \sqsubseteq w$ in $\mathcal{L}_{v'}$, and no $w' \sqsubset w$ is in $\mathcal{L}_v$, and by checking that $C_w \subseteq C_v$. Hence $\mathcal{L}_v$s are decidable uniformly in v.

Let $h : \mathbb{N}^* \to [0,\infty)$ be defined by

$$h(v) = \sum_{w \in \mathcal{L}_v} (\log_2 d(w) + c + 1) \frac{\mu(w)}{\gamma(v)}.$$

Since $d \geq 2^{-c}$, it follows that h is a positive real-valued function.

We know that $\log_2 d + c + 1$ is a supermartingale by Lemma 26. We have

$$\sum_{i \in \mathbb{N}} h(vi)\gamma(vi) = \sum_{i \in \mathbb{N}} \sum_{w \in \mathcal{L}_{vi}} (\log_2 d(w) + c + 1)\mu(w) \leq \sum_{u \in \mathcal{L}_v} \sum_{\substack{i \in \mathbb{N}, \\ w \in \mathcal{L}_{vi}, \\ u \sqsubseteq w}} (\log_2 d(w) + c + 1)\mu(w).$$

Since $\mathcal{L}_{vi}$ is a prefix-free set for each $i \in \mathbb{N}$, by the Kolmogorov inequality [19], the above is at most $\sum_{u \in \mathcal{L}_v} (\log_2 d(u)+c+1)\mu(u)$, which is $h(v)\gamma(v)$, establishing that h is a supermartingale.

Suppose the savings account function of the $\log_2 d + c + 1$ supermartingale is denoted s_d. Then for every $D \in \Sigma^\infty$ and every $n \in \mathbb{N}$, we have $s_d(D \upharpoonright n) \leq s_d(D \upharpoonright n+1)$ and that $\lim_{n\to\infty} s_d(B \upharpoonright n) = \infty$. If $s_d(u) \geq M > 0$, where C_u is the smallest cylinder which covers C_v, $v \in \mathbb{N}^*$, then we have

$$h(v) \geq \sum_{w \in \mathcal{L}_v} s_d(w) \frac{\mu(w)}{\gamma(v)} \geq \frac{M}{\gamma(v)} \sum_{w \in \mathcal{L}(v)} \mu(w) = \frac{M\mu(v)}{\gamma(v)},$$

where the equality follows by Lemma 27. By Lemma 13, similar to the argument of the converse direction, we conclude that the above quantity is at least $M \ln(2)$. It follows that $X \in S^{\infty}_{\text{str}}[d]$.

If d is lower semicomputable, then so is $(\log_2 d + c + 1)$. Since $\mathcal{L}_v$ is decidable uniformly in v, it follows that h is the sum of a computably enumerable sequence of lower semicomputable terms, hence is lower semicomputable.

If d is computable, then so is $(\log_2 d + c + 1)$, witnessed by, say, $\hat{\ell}_d : \mathbb{N}^* \times \mathbb{N} \to [0,\infty) \cap \mathbb{Q}$. For each $v \in \mathbb{N}^*$, let $\langle w_{v,j} \rangle_{j \in \mathbb{N}}$ be a computable enumeration of $\mathcal{L}_v$ in increasing order, which exists since $\mathcal{L}_v$ is decidable. Hence, $\hat{h} : \mathbb{N}^* \times \mathbb{N} \to [0,\infty) \cap \mathbb{Q}$ defined below witnesses the computability of h. For $v \in \mathbb{N}^*$ and $n \in \mathbb{N}$, define

$$\hat{h}(v,n) = \sum_{j=1}^{N_{v,n}} \hat{\ell}_d(w_{v,j}) \frac{\mu(w_{v,j})}{\hat{\gamma}(v,n)},$$

where

$$N_{n,v} = \min\left\{ m \in \mathbb{N} \mid \sum_{j=1}^{m} \mu(w_{v,j}) \ > \ \mu(vi) - 2^{-n} \right\}.$$

Then, $N_{n,v}$ exists for all n and v by Lemma 27. Moreover, $N_{n,v}$ is computable uniformly in n and v. We now show that for all n, $|\hat{h}(v,n) - h(v)| \leq (2 + c + 1)2^{-n}$, showing that h is computable.

For any $w \in \Sigma^*$, we know that $d(w) \leq 2^{|w|}$, hence $\log_2 d(w) + c + 1 \leq |w| + c + 1$. Further, $\sum_{j=N_n+1}^{\infty} \mu(w_{v,j}) \leq 2^{-n}$. Hence,

$$\sum_{j=N_n+1}^{\infty} \frac{\log_2 d(w_{v,j}) + c + 1}{2^{|w_{v,j}|}} \leq \sum_{j=N_n+1}^{\infty} \frac{|w_{v,j}| + c + 1}{2^{|w_{v,j}|}},$$

which, by Lemma 28, is upper bounded by a term computable from n and decreasing to 0 as $n \to \infty$. It follows that h is computable. ◀

7 Effective dimension of continued fractions using s-gales

Adapting the approach of Lutz [14], Lutz and Mayordomo [15] for finite alphabets, we define effective Hausdorff dimension of sets of continued fractions.

▶ **Definition 16.** *Let $s \in [0,\infty)$ and $\mathbb{N}^\infty$ denote the set of infinite sequences of positive integers.*

- *A continued fraction s-gale is a function $d : \mathbb{N}^* \longrightarrow [0,\infty)$ that satisfies the condition*

$$d(w)[\gamma(C_w)]^s = \sum_{i \in \mathbb{N}} d(wi)[\gamma(C_{wi})]^s$$

for all $w \in \mathbb{N}^*$.

- *We say that d succeeds on a sequence $Q \in \mathbb{N}^\infty$ if $\limsup_{n\to\infty} d(Q \upharpoonright n) = \infty$.*
- *The success set of d is $S^\infty(d) = \{Q \in \mathbb{N}^\infty \mid d \text{ succeeds on } Q\}$.*
- *For $\mathcal{X} \subseteq \mathbb{N}^\infty, \mathcal{G}(\mathcal{X})$ denotes the set of all $s \in [0,\infty)$ such that for every $X \in \mathcal{X}$, there exists a lower semicomputable continued fraction s-gale d which succeeds on X.*
- *The effective Hausdorff dimension of a set $\mathcal{S} \subseteq \mathbb{N}^\infty$ is the infimum of the set $\mathcal{G}(X)$.*

It is possible to view s-gales as martingales with a specified rate of success. First, we show that an s-gale can be converted into a martingale by multiplying the capital of the s-gale with an adjusted rate for the success. This is similar to the corresponding result for binary s-gales and martingales in [14].

▶ **Lemma 17.** *Let $d : \mathbb{N}^* \to [0,\infty)$ be an s-gale. Then $g : \mathbb{N}^* \to [0,\infty)$ defined by $g(v) = d(v)\gamma^{s-1}(v)$ is a continued fraction martingale.*

Proof. It is clear that $g(\lambda) = 1$. Further, for $v \in \mathbb{N}^*$, we have

$$\sum_{i\in\mathbb{N}} g(vi)\gamma(vi) = \sum_{i\in\mathbb{N}} d(vi)\gamma^{s-1}(C_{vi})\gamma(vi) = \sum_{i\in\mathbb{N}} d(vi)\gamma^{s}(vi) = d(v)\gamma^{s}(v) = g(v)\gamma(v),$$

where the penultimate equality follows since d is an s-gale. ◀

The following helps us to relate the success rate of martingales to the dimension.

▶ **Lemma 18.** *Let $d : \mathbb{N}^* \to [0, \infty)$ be a lower semicomputable continued fraction martingale, and $s \in (0, 1)$. If $X \in \mathbb{N}^\infty$ has infinitely many prefix lengths n for which*

$$d(X \upharpoonright n) \geq \gamma^{s-1}(X \upharpoonright n),$$

then $\dim(X) \leq s$.

Thus, we have the following characterization of dimension of continued fractions in terms of the success rate of martingales.

▶ **Theorem 19.** *For any $X \in \mathbb{N}^\infty$, $s \in (0, 1)$, we have $\dim(X) \leq s$ if and only if there is a continued fraction martingale $d : \mathbb{N}^* \to [0, \infty)$ such that for infinitely many n, $d(X \upharpoonright n) \geq \gamma^{s-1}(C_{X\upharpoonright n})$.*

8 Continued fractions with dimension 0 and computability

▶ **Lemma 20.** *Every computable continued fraction has effective dimension zero.*

Proof. Let $X = [0; a_1, a_2, \dots]$ be an arbitrary continued fraction such that $a_i \in \mathbb{N}$. Let M be total computable function on $\mathbb{N}$ such that for all $i \in \mathbb{N}$, $M(i) = a_i$.

Consider the function $d : \mathbb{N}^* \to [0, \infty)$ which bets all of current capital along the sequence computed by M, defined by $d(a_1, a_2, \dots a_n) = \gamma^{-s}(C_{a_1,a_2,\dots a_n})$. Let $d(v) = 0$ if v is not a prefix of X.

Then d is an s-gale, since for every $v \in \mathbb{N}^*$ which is a prefix of S,

$$\sum_{i\in\mathbb{N}} d(vi)\gamma^s(C_{vi}) \;=\; \frac{\gamma^s(C_{vM(|v|)})}{\gamma^s(C_{vM(|v|)})} \;=\; 1 \;=\; \gamma^{-s}(C_v)\gamma^s(C_v) \;=\; d(v)\gamma^s(C_v).$$

For $v \in \mathbb{N}^*$ which is not a prefix of X, $d(v) = 0$, hence $\sum_{i\in\mathbb{N}} d(vi)\gamma^s(C_{vi}) = 0 = d(v)\gamma^s(C_v)$.

Since $\gamma([0; a_1, \dots, a_n]) \to 0$ as $n \to \infty$ and $s > 0$, it follows that $\gamma^{-s}([0; a_1, \dots, a_n]) \to \infty$ as $n \to \infty$. Hence $X \in S^\infty[d]$. Since s was arbitrary, the infimum of all s such that there is an s-gale which succeeds on X is 0. ◀

However, the converse does not hold in general. We show that there are uncomputable continued fractions with dimension 0.

The standard technique for binary sequences uses the notion of “dilution” - we add a few bits from a Martin-Löf random sequence, and intersperse it with a large number of 0s. By making the number of zeroes grow in an unbounded manner, we can construct a dimension 0 sequence.

Surprisingly, with continued fractions, we can perform this “dilution” by following every “random” integer with a *single* integer. We do not require arbitrarily long computable stretches. We are able to do this since the underlying alphabet is infinite.

To make the continued fraction uncomputable, at every odd location, we copy the integer from a Martin-Löf random continued fraction. To make the continued fraction have dimension

0, at every even location, we computably choose a large integer so that an s-gale can make unbounded amounts of money by betting.

The construction is involved, because the underlying probability measure, Gauss measure, is not a product distribution. Hence the choice of these "large integers" at even locations necessarily depend on the previous integers. The argument which follows uses several approximation techniques.

▶ **Lemma 21.** *There is an uncomputable continued fraction with dimension 0.*

Proof. Let Y be a Martin-Löf random continued fraction. Let X be the continued fraction defined by

$$X[n] = \begin{cases} Y_{\lceil n/2 \rceil} & \text{if } n \text{ is odd,} \\ f(X \upharpoonright n-1) & \text{otherwise,} \end{cases}$$

where $f : \mathbb{N}^* \to \mathbb{N}$ defined by $f(v) = [\max(v)+2]^{(|v|)^2}$ for $v \in \mathbb{N}^*$. We show that $\dim_\gamma(X) = 0$. It suffices to show that for all $s \in (0,1)$, there is an s-gale that succeeds on X.

Consider the computable function $d : \mathbb{N}^* \to [0,\infty)$ defined by $d(\lambda) = 1$ and for every v of odd length and $i \in \mathbb{N}$, letting $d(vi) = d(v)\gamma^{1-s}(C_{vi|v})$. For every v of even length, $j = f(v)$, let $d(vj) = d(v)\gamma^{-s}(C_{vj|v})$, and for $k \neq f(v)$, let $d(vk) = 0$.

If $|v|$ is odd, then

$$\sum_{i \in \mathbb{N}} d(vi)\gamma^s(vi|v) = d(v) \sum_{i \in \mathbb{N}} \gamma^{1-s}(vi|v)\gamma^s(vi|v) = d(v) \sum_{i \in \mathbb{N}} \gamma(vi|v) = d(v),$$

and if $|v|$ is even, then letting $j = f(v)$,

$$\sum_{i \in \mathbb{N}} d(vi)\gamma^s(vi|v) = d(v)\frac{\gamma^s(vj|v)}{\gamma^s(vj|v)} = d(v).$$

Hence d is an s-gale.

We show now that $X \in S^\infty[d]$. Denote $X \upharpoonright 2k-1$ by v. Let $X[2k] = Y[k]$ be denoted by i and $X[2k+1] = f(vi)$ be denoted by j. Then

$$\frac{d(vij)}{d(v)} = \frac{1}{\gamma^{s-1}(vi|v)\gamma^s(vij|vi)} = \frac{\gamma(vi|v)}{\gamma^s(vi|v)\gamma^s(vij|vi)} \geq \frac{\gamma(vi|v)}{\gamma^s(vij|vi)},$$

since $0 \leq \gamma^s(vi|v) \leq 1$. By Lemma 2, it follows that

$$\frac{\gamma(vi|v)}{\gamma^s(vij|vi)} \geq \frac{\mu(vi|v)}{2(\ln 2)^{1-s}\mu^s(vij|vi)}.$$

We have that $\mu(vi|v)$ is

$$\frac{q_{2k-1}(q_{2k-1}+q_{2k-2})}{q_{2k}(q_{2k}+q_{2k-1})} \geq \frac{q_{2k-1}^2}{2q_{2k}^2} = \left(\frac{q_{2k-1}^2}{2(iq_{2k-1}+q_{2k-2})^2}\right) \geq \left(\frac{q_{2k-1}^2}{2(i+1)^2 q_{2k-1}^2}\right) = \frac{1}{2(i+1)^2} \geq \frac{1}{2(m+2)^2},$$

where $m = \max(vi)$. Similarly

$$\frac{1}{\mu(vij|vi)} = \frac{q_{2k+1}(q_{2k+1}+q_{2k})}{q_{2k}(q_{2k}+q_{2k-1})} \geq \frac{q_{2k+1}}{q_{2k}+q_{2k-1}} = \frac{jq_{2k}+q_{2k-1}}{q_{2k}+q_{2k-1}} \geq \frac{jq_{2k}+q_{2k-1}}{2q_{2k}} \geq \frac{j}{2}$$

Since $j = (m+2)^{4k^2}$, it follows that

$$\frac{\mu(vi|v)}{2(\ln 2)^{1-s}\mu^s(vij|vi)} \geq \frac{1}{2(m+2)^2}\frac{(m+2)^{4k^2 s}}{2^{s+1}(\ln 2)^{1-s}} = \frac{(m+2)^{4k^2 s-2}}{2^{s+2}(\ln 2)^{1-s}}$$

For fixed s, as $k \to \infty$, the above quantity is greater than 2. It follows that d succeeds on X. Since $s \in (0,1)$ was arbitrary, we can conclude that $\dim_\gamma(X) = 0$. ◀

9 Continued fractions with dimension 1 and Martin-Löf randomness

In this section, we study the relationship between Martin-Löf randomness of continued fractions, normality of continued fractions, and the notion of effective dimension 1. We show that all Martin-Löf random continued fractions have effective dimension 1. However, there are continued fractions with effective dimension 1, which are normal as well, but which are not Martin-Löf random.

▶ **Lemma 22.** *Every Martin-Löf random continued fraction has effective dimension 1.*

Proof. Let $Y \in \mathbb{N}^\infty$ have $s = \dim(Y) \leq 1$. Let $d : \mathbb{N}^* \to [0, \infty)$ be a lower semicomputable s-gale that succeeds on Y. Consider the lower semicomputable function $h : \mathbb{N}^* \to [0, \infty)$ defined by $h(v) = d(v)\gamma^{s-1}(C_v)$, for $v \in \mathbb{N}^*$. Then

$$\sum_{i \in \mathbb{N}} h(vi)\gamma(C_{vi}) = \sum_{i \in \mathbb{N}} d(vi)\gamma^s(C_{vi}) = d(v)\gamma^s(C_v) = h(v)\gamma(C_v),$$

where the second last equality follows by the fact that d is an s-gale.

Suppose $d(Y \upharpoonright n) > M$. Then $h(Y \upharpoonright n) > M\gamma^{s-1}(Y \upharpoonright n) > M$. Since $Y \in S^\infty[d]$, it follows that $Y \in S^\infty[h]$. Hence Y is not a Martin-Löf random continued fraction. ◀

However, there are sequences with c.e. dimension 1, which are not random. The idea is to intersperse the integer "1" at computable locations which are spaced very sparsely apart. The proof that the resulting number is not Martin-Löf random uses the following estimate on conditional Gauss probabilities, which, to our knowledge, is not present in literature.

▶ **Lemma 23.** *For any $v = [0; v_1, \ldots, v_n] \in \mathbb{N}^*$, we have*

$$\frac{1}{2\ln(2)(2v_n + 3)} \leq \gamma(C_{v1|v}) \leq \frac{1}{2\ln(2)}.$$

The above lemma shows that the conditional probability of 1 in any cylinder $[0; v_1, \ldots, v_n, 1]$ can be arbitrarily small if v_n is arbitrarily large. Hence a betting function to win arbitrarily large amounts. In the following constructions in the paper, unlike in the dimension 0 construction, it becomes necessary to allow a betting function to win, but also to prevent large wins, at specific positions. We control this winning amount by inserting 1s at computable locations only when v_n is *bounded.*

▶ **Lemma 24.** *There is a continued fraction with effective dimension 1, which is normal, but which is not Martin-Löf random.*

Proof. Let Y be a Martin-Löf random continued fraction. We construct $X \in \mathbb{N}^\infty$ in stages, as follows.

At each stage $s \geq 1$, we copy at least $s!$ integers from Y into X, maintaining the relative order. Associated with each stage, we keep a cumulative count N_s of the number of integers we have copied from Y, in stages 1 through s inclusive.

Construction. At stage 1, we set $X[i] = Y[i]$ starting from $i = 1$, until we see a position with $Y[i] = 1$. We denote this position as N_1. Such a position always exists since Y is Martin-Löf random by Theorem 11. Set $X[N_1 + 1] = 1$.

Note that at every stage, we insert exactly one 1 into X, which is not present in Y.

At stage $s > 1$, we proceed as follows. Note that X is longer than Y by exactly $s - 1$ digits at the start of stage s. Set $X[N_{s-1} + (s-1) + j] = Y[N_{s-1} + j]$, for j from 1 through at least $s!$, and until we encounter a position in Y which has a 1. Such a position exists by the normality of Y. We denote this position as K_s, and let $N_s = N_{s-1} + K_s$. Set $X[N_s + (s-1) + 1] = 1$.

Let P_X be the set of positions where we have inserted ones into X, and P_Y be the set of positions in Y after which we have inserted ones in X while copying. At each stage s, we copy at least $s!$ entries from Y before inserting the additional 1 into X. Note that P_Y is computable from Y. Hence for all sufficiently large n, the number of entries in P_X and P_Y which are less than or equal to n is $o(\log n)$. (End of construction)

Verification. We now show that there is a lower semicomputable martingale $d : \mathbb{N}^* \to [0, \infty)$ which succeeds on X, showing that X is not Martin-Löf random. Let $d(\lambda) = 1$, and for every $v \in \mathbb{N}^*$, if $|v| + 1 \notin P_X$, then $d(vi) = d(v)$. It is clear that on these $v \in \mathbb{N}^*$, the martingale condition is satisfied. If $|v| + 1 \in P_X$, then let $d(v1) = d(v)\gamma^{-1}(C_{v1|v})$, and $d(vj) = 0$ for all $j \neq 1$. For such $v \in \mathbb{N}^*$, we have

$$\sum_{i \in \mathbb{N}} d(vi)\gamma(C_{vi|v}) = d(v1)\gamma(C_{v1|v}) = d(v)\frac{\gamma(C_{v1|v})}{\gamma(C_{v1|v})} = d(v),$$

proving that d is a martingale. Since checking for membership in P is computable based on the prefix v, it follows that d is lower semicomputable.

To see that d succeeds on X, we observe that at every position in P, d multiplies its previous capital by $\gamma^{-1}(C_{v1|v})$, and on other prefixes of X, d preserves its capital. By Lemma 23, $\gamma^{-1}(C_{v1|v}) \geq 2 \ln 2$. Thus, $\lim_{n \to \infty} d(X \upharpoonright n) = \infty$.

We now show that if $\dim(X) < 1$, then Y is not Martin-Löf random. Let $s \in (0, 1)$ and $h : \mathbb{N}^* \to [0, \infty)$ be a lower semicomputable s-gale which succeeds on X. At positions $n \in P_X$, we can assume without loss of generality that

$$h(X \upharpoonright n) = h(X \upharpoonright (n-1)) \quad \gamma^{-s}((X \upharpoonright (n-1))1 \mid (X \upharpoonright (n-1))), \tag{4}$$

i.e. h attains the maximum possible capital on the positions in P_X.

Construct a martingale $g : \mathbb{N}^* \to [0, \infty)$ thus. Let $g(\lambda) = 1$. If $v \in \mathbb{N}^*$ is such that $|v| \notin P_Y$, then for every $i \in \mathbb{N}$, let $g(vi) = h(vi)\gamma^{s-1}(vi)$. Otherwise, let $g(vi) = h(v1i)\gamma^s(v1|v)\gamma^{s-1}(vi)$.

If v belongs to the first case above, then

$$\sum_{i \in \mathbb{N}} g(vi)\gamma(vi) = \sum_{i \in \mathbb{N}} h(vi)\gamma^{s-1}(vi)\gamma(vi) = \sum_{i \in \mathbb{N}} h(vi)\gamma^s(vi) = h(v)\gamma^s(v) = g(v)\gamma(v),$$

and otherwise,

$$\sum_{i \in \mathbb{N}} g(vi)\gamma(vi) = \sum_{i \in \mathbb{N}} h(v1i)\gamma^s(v1|v)\gamma^s(vi) = h(v1)\gamma^s(v1|v)\gamma^s(v) = h(v)\gamma^s(v) = g(v)\gamma(v),$$

where the second equality follows since h is an s-gale, and the third inequality follows by (4). Hence, g is a lower semicomputable martingale.

By Lemma 1 and 2, $\gamma^{s-1}(vi) > 2^{(1-s)|vi|}(\ln 2)^{1-s}$. Recall that P_Y contains $o(\log n)$ elements which are less than n. Since every position in P_X is preceded by $v_n = 1$, it follows that $\gamma^s(v1|v) \geq 1/(10 \ln(2))$ for every v with $|v| \in P_Y$. Hence $g(Y \upharpoonright n) \geq \frac{2^{(1-s)n}(\ln 2)^{1-s}}{n}$ which tends to ∞ as $n \to \infty$. Hence Y is not Martin-Löf random, which is a contradiction.

Since s is arbitrary, it follows that $\dim(X) = 1$. ◀

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Appendix

Proof of Lemma 1. We know that Lebesgue measure of C'_p(cylinder set of the binary expansion) is equal to $\frac{1}{2^n}$ where $|p| = n$. We now prove by mathematical induction on n that,

$$\mu(C_{[0;a_1,a_2\dots a_n]}) \ \leq \ \frac{1}{2^n}$$

- Base case : $\mu(C_{[0;a_1]}) = \frac{1}{a_1(a_1+1)}$ which is strictly decreasing in a_1. The maximum occurs at $a_1 = 1$, where $\mu(C_1) = \frac{1}{2}$, as required.
- Inductive step : We assume that the above claim is true till some k.

$$\mu(C_{[0;a_1,a_2\dots a_k]}) \ \leq \ \frac{1}{2^k}$$

Now, assume $\frac{p_k}{q_k}$ is the k^{th} convergent of $[0; a_1, a_2 \dots]$.

$$\mu(C_{[0;a_1,a_2\dots a_k]}) = \left|\frac{p_k}{q_k} - \frac{p_k + p_{k-1}}{q_k + q_{k-1}}\right| = \frac{p_k q_{k-1} - q_k p_{k-1}}{q_k(q_k + q_{k-1})} \leq \frac{1}{2^k} \tag{5}$$

We show that

$$\mu(C_{[0;a_1,a_2\dots a_k,a_{k+1}]}) \ \leq \ \frac{1}{2^{k+1}}$$

We have

$$\begin{aligned}\mu(C_{a_1,a_2\dots a_k,a_{k+1}}) &= \left|\frac{a_{k+1}p_k + p_{k-1}}{a_{k+1}q_k + q_{k-1}} - \frac{(a_{k+1}+1)p_k + p_{k-1}}{(a_{k+1}+1)q_k + q_{k-1}}\right| \\ &= \left|\frac{p_{k-1}q_k - p_k q_{k-1}}{(a_{k+1}q_k + q_{k-1})(q_k q_{k+1} + q_k + q_{k-1})}\right|\end{aligned}$$

By multiplying and dividing on numerator and denominator with $q_k(q_k + q_{k-1})$ we get,

$$\mu(C_{a_1,a_2\dots a_k,a_{k+1}}) = \left|\frac{p_{k-1}q_k - p_k q_{k-1}}{q_k(q_k + q_{k-1})}\right| \left|\frac{q_k(q_k + q_{k-1})}{(a_{k+1}q_k + q_{k-1})(q_k q_{k+1} + q_k + q_{k-1})}\right|$$

From our assumption, equation (5) signifies that left term is less than or equal to $\frac{1}{2^k}$. We show that the right term is less than $\frac{1}{2}$. We have

$$\left|\frac{q_k(q_k + q_{k-1})}{(a_{k+1}q_k + q_{k-1})(q_k q_{k+1} + q_k + q_{k-1})}\right| = \left|\frac{(q_k + q_{k-1})}{(a_{k+1}q_k + q_{k-1})(a_{k+1} + 1 + \frac{q_{k-1}}{q_k})}\right|.$$

The above term is less than $\frac{1}{2}$ by the fact that a_{k+1}, q_k, q_{k-1} are always greater than or equal to 1, thus establishing the result.

◀

Proof of Lemma 2. For any interval B,

$$\gamma(B) = \frac{1}{\ln 2} \int_B \frac{1}{1+x}\, dx.$$

Since $0 \leq x \leq 1$, we know that $0.5 \leq \frac{1}{1+x} \leq 1$. By the definition of Lebesgue measure, we have $\mu(B) = \int_B dx$. Hence, we have

$$\frac{1}{2\ln 2}\mu(B) \leq \gamma(B) \leq \frac{1}{\ln 2}\mu(B).$$

◀

Proof of Lemma 5. We prove the result by induction on k. Initially, assume that $k = |v|+1$. Then, $\sum_{w \in S} d(w)\gamma(w) \leq \sum_{i \in N} d(vi)\gamma(vi) \leq d(v)\gamma(v)$ since d is a supermartingale. Suppose the claim holds when strings in S have length at most k, and we show that the claim holds when strings in S have length at most $k+1$.

Let $w \sqsupseteq v$ and $|w| = k$. Then

$$\sum_{\substack{w' \in S \\ w' \sqsupset w}} d(w')\gamma(w') \leq d(w)\gamma(w),$$

by the inductive hypothesis. Hence $\sum_{w' \in S} d(w')\gamma(w')$ can be upper bounded by

$$\sum_{\substack{v \sqsubseteq w \sqsubseteq w' \\ |w|=k}} d(w)\gamma(w),$$

which by the inductive assumption, is at most $d(v)\gamma(v)$. ◀

Proof of Theorem 8. Let $d_1, d_2, \ldots$ be the martingales as given. Now consider the martingale d such that $d(w) = \sum_{i=1}^{\infty} d_i(w)2^{-i}$, for any $w \in \mathbb{N}^*$.

We now prove that d is a martingale. Since $d_i(\lambda) = 1$ for every $i = 1, 2, \ldots$, it is clear that $d(\lambda) = 1$. We have

$$d(w)\gamma(C_w) = \left[\sum_{i=1}^{\infty} \frac{d_i(w)}{2^i}\right]\gamma(C_w) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left[\frac{d_i(wj)}{2^i}\right]\gamma(C_{wj})$$

since d_i, $i = 1, 2, \ldots$, are martingales. Thus, we have $d(w).\gamma(C_w) = \sum_{j=1}^{\infty} d(wj).\gamma(C_{wj})$. It follows that d is a martingale.

For each martingale d_i, $i = 1, 2, \ldots$, let $\hat{d}_i : \mathbb{N}^* \times \mathbb{N} \to \mathbb{Q} \cap [0, \infty)$ be a function witnessing its lower semicomputablility. Then, the function $\hat{d} : \mathbb{N}^* \times \mathbb{N} \to \mathbb{Q} \cap [0, \infty)$ defined by $\hat{d}(w, n) = \sum_{i=1}^{\infty} \hat{d}_i(w, n)2^{-i}$, for any $w \in \mathbb{N}^*$ and $n \in \mathbb{N}$, witnesses the lower semicomputability of d.

Let $X \in S^{\infty}[d_i]$ (or, alternatively, $X \in S^{\infty}_{\text{str}}[d]$). Assume that on some prefix $X \upharpoonright n$, we have $d_i(X \upharpoonright n) \geq M$, for some $M > 0$ and positive integer i. Since d_j, $j \neq i$, are non-negative functions, $d(X \upharpoonright n) \geq d_i(X \upharpoonright n)/2 > M/2^i$. Note that the multiplication factor $\frac{1}{2^i}$ depends only on d_i and not on either M or X. Hence we conclude that if $\limsup_{n\to\infty} d_1(X \upharpoonright n) = \infty$, then $\limsup_{n\to\infty} d(X \upharpoonright n) = \infty$, and if $\liminf_{n\to\infty} d_1(X \upharpoonright n) = \infty$, then $\liminf_{n\to\infty} d(X \upharpoonright n) = \infty$. ◀

Proof of Theorem 10. Let d be a lower semicomputable supermartingale which succeeds on X. Define, for all integers $n \geq 1$, the function $g_n : \mathbb{N}^* \to [0, 1]$ as follows. Let $g_n(\lambda) = \frac{d(\lambda)}{2^n}$.

For $v \in \mathbb{N}^*$ and $i \in \mathbb{N}$, if $g_n(v) \geq 1$, then let $g_n(vi) = 1$. Otherwise, let $g(vi) = \min\{\frac{d(vi)}{2^n}, 1\}$.

Let $v \in \mathbb{N}^*$ satisfy $g_n(v) \geq 1$. Then for every $i \in \mathbb{N}$, $g_n(vi) = 1$, and we have

$$\sum_{i \in \mathbb{N}} g_n(vi)\gamma(i|v) = \sum_{i \in \mathbb{N}} \gamma(i|v) = 1 \leq g_n(v)$$

hence the supermartingale condition holds at v. Otherwise, we have $g_n(v) = d(v)2^{-n} < 1$, and thus $g_n(vi) = \min\{d(vi)2^{-n}, 1\}$. Hence,

$$\sum_{i\in\mathbb{N}} g_n(vi)\gamma(i|v) \leq 2^{-n}\sum_{i\in\mathbb{N}} d(vi)\gamma(i|v) \leq 2^{-n}d(v) = g_n(v),$$

establishing that g_n is a supermartingale.

Let $\hat{d} : \mathbb{N}^* \times \mathbb{N} \to [0,\infty)$ witness the lower semicomputability of d. Then $\hat{g}_n : \mathbb{N}^* \times \mathbb{N} \to [0,\infty)$ defined below, witnesses the lower semicomputability of g_n. Let $\hat{g}_n(\lambda, m) = \frac{\hat{d}(\lambda,m)}{2^n}$ for all m. For $v \in \mathbb{N}^*$, and $i, m \in \mathbb{N}$, define

$$\hat{g}_n(vi, m) = \begin{cases} \min\{\hat{d}(vi, m)2^{-n}, 1\} & \text{if } \hat{g}_n(v, m) < 1 \\ 1 & \text{otherwise.} \end{cases}$$

We show that for every v, i and m, $\hat{g}_n(vi, m) \leq \hat{g}_n(vi, m+1) \leq g_n(vi, m)$.

Fix $v \in \mathbb{N}^*$ and $i \in \mathbb{N}$. If for all $m \in \mathbb{N}$, the computation above falls entirely within the first case, or entirely within the second case, then the monotonicity of $\hat{g}$ follows from the monotonocity of $\hat{d}$.

Now suppose that for finitely many m, the computation falls into the first case, and for all sufficiently large m, the second case applies. This implies that $g(v) \geq 1$. Hence $g_n(vi) = 1$. Then for all m, $\hat{g}_n(vi, m) \leq 1 = g_n(vi)$. Further, by the monotonicity of $\hat{d}$, we also have $\hat{g}_n(vi, m) \leq \hat{g}_n(vi, m+1)$.

To see that convergence holds, first observe that if $g_n(v) \geq 1$, then the second case applies for all sufficiently large m, whence we have $\hat{g}_n(vi, m) = 1$, which is the value of $g_n(vi)$. Suppose, otherwise, that $g_n(v) < 1$. The first case always applies, and the convergence of $\hat{d}$ implies that the computation converges to $\min\{d(vi)2^{-n}, 1\}$, as required.

Define the function $g = \sum_{n=1}^{\infty} g_n$. Then g is a lowersemicomputable supermartingale. Since d succeeds on X, for any M, there is an $n \in \mathbb{N}$ on which $d(X \upharpoonright n) \geq 2^M$. Hence, for all $n' \geq n$, $g_1(X \upharpoonright n')$, ..., $g_M(X \upharpoonright n') \geq 1$, implying that for all $n' \geq n$, $g(X \upharpoonright n') \geq M$. It follows that $\liminf_{n'\to\infty} g(X \upharpoonright n') = \infty$, as required.

Now, suppose d is a computable supermartingale which succeeds on X. Then we define computable functions g, h and s: $\mathbb{N}^* \times \mathbb{N} \to [0,\infty)$ such that $g = h + s$, and g is a supermartingale which strongly succeeds on X, with $g \geq s$ and s monotone increasing over prefix lengths.

We define h by initially letting $h(\lambda) = 1$. Associated with each string v, we keep an integer m_v. Initially, $m_\lambda = 2$. For an arbitrary $v \in \mathbb{N}^*$, $i \in \mathbb{N}$, we let

$$h(vi) = \frac{d(vi)}{d(v)}h(v)$$

if this amount is at most 2^{m_v}, and let $s(vi) = s(v)$. Otherwise

$$h(vi) = \frac{d(vi)}{d(v)}h(v) - 1,$$

let $s(vi) = s(v) + 1$ and let $m_{vi} = m_v + 1$.

Let vi, $v \in \mathbb{N}^*$, $i \in \mathbb{N}$ be a string where the second case applies. Then

$$h(vi) + s(vi) = \frac{d(vi)}{d(v)}h(v) - 1 + s(v) + 1 = \frac{d(vi)}{d(v)}h(v) + s(v),$$

and this is identical to the value when the first case applies. Hence, we have

$$\sum_{i\in\mathbb{N}} g(vi)\gamma(i|v) \quad = \quad \left[\sum_{i\in\mathbb{N}} d(vi)\gamma(i|v)\right]\frac{h(v)}{d(v)} + s(v)\sum_{i\in\mathbb{N}}\gamma(i|v) \quad \leq \quad h(v)+s(v)=g(v),$$

since d is a supermartingale. Thus g is a supermartingale.

Since division and subtraction are computable, and d is computable, it follows that g, s and h are computable. Moreover, if there is an n for which $d(X \upharpoonright n) \geq 2^M$, then for all $n' > n$, $s(X \upharpoonright n') \geq M$. Since $\limsup_{n\to\infty} d(X \upharpoonright n) = \infty$, we conclude that $\liminf_{n\to\infty} g(X \upharpoonright n) \geq \liminf s(X \upharpoonright n) = \infty$. ◀

Proof of Theorem 11. Let $X \in \mathbb{N}^\infty$ and m be the least positive integer that appears only finitely often in X. Let $m \neq X_k$ for $k \geq K_0$. Consider $d : \mathbb{N}^* \to [0,\infty)$ defined by $d(\lambda) = 1$, and for arbitrary strings as follows. For $v \in \mathbb{N}^*$ with $|v| < K_0$, for every $i \in \mathbb{N}$, let $d(vi) = 1$. For $v \geq K_0$, and $i \in \mathbb{N}$, let

$$d(vi) = \begin{cases} \frac{d(v)}{1-\gamma(vm|v)} & \text{if } i \neq m \\ 0 & \text{otherwise.} \end{cases}$$

If $|v| < K_0$, for every $i \in \mathbb{N}$, $\sum_{i\in\mathbb{N}} d(vi)\gamma(vi|v) = 1 = d(v)$. For $|v| \geq K_0$, then we have

$$\sum_{i\in\mathbb{N}} d(vi)\gamma(vi|v) = d(v)\frac{1-\gamma(vm|v)}{1-\gamma(vm|v)} = d(v),$$

establishing that d is a martingale. It is clear that d is computable since $\gamma(vm|v) \neq 0$ and γ is a computable probability measure.

For sufficiently large n,

$$d(X \upharpoonright n) \geq \prod_{i=K_0+1}^{n} \frac{1}{1-\gamma((X \upharpoonright i)m \mid (X \upharpoonright i))}.$$

We lower bound $\gamma(vm|v)$ over $v \in \mathbb{N}^*$ as follows. Let $v = [0; v_1, \ldots, v_n]$. Let $\frac{p_n}{q_n} = v$ and $\frac{p_{n-1}}{q_{n-1}} = [0; v_1, \ldots, v_{n-1}]$. Then we have

$$\frac{\mu(vm)}{\mu(v)} = \frac{q_n(q_n+q_{n-1})}{q_{n+1}(q_{n+1}+q_n)} > \frac{q_n^2}{2q_{n+1}^2} = \frac{q_n^2}{2(mq_n+q_{n-1})^2} \geq \frac{q_n^2}{2(m+1)^2q_n^2} = \frac{1}{2(m+1)^2}.$$

Hence,

$$\gamma(vm|v) \geq \frac{1}{4\ln 2(m+1)^2} = c,$$

say. Then $0 < c < 1$.

We have $d(X \upharpoonright n) \geq (1-c)^{-n+K_0}$. Hence $X \in S^\infty_{\text{str}}[d]$. ◀

Proof of Lemma 12. Let $j = \lfloor -\log_2(b-a)\rfloor + 1$. We know that

$$-\log_2(b-a) \leq j \leq -\log_2(b-a)+1,$$

hence $(b-a) \geq 2^{-j} \geq (b-a)/2$. It follows that exactly dyadic rational of the form $m/2^j$, $0 \leq m < 2^j$ is in (a,b).

It follows that four dyadic intervals of length $\frac{1}{2^j}$ cover the interval $[a,b]$. ◀

Proof of Lemma 13. Let j be the smallest integer such that $\frac{1}{2^j} \leq (b-a)$. By the proof of the previous lemma, $\frac{1}{2^j} \geq (b-a)/2$.

Hence there is some dyadic interval $(k/2^{j+1}, (k+1)/2^{j+1})$ which is a subinterval of $[a, b)$. Since $\frac{1}{2^j} \geq (b-a)/2$, we have $1/2^{j+1} \geq \frac{1}{4}(b-a)$. ◀

Proof of Lemma 18. Let d be a martingale, $s \in (0,1)$ and s' be an arbitrary real such that $s < s' < 1$. It suffices to show that an s'-gale $d' : \mathbb{N}^* \to [0, \infty)$ succeeds on X. Define, for every $v \in \mathbb{N}^*$, $d'(v) = d(v)\gamma^{1-s'}(v)$. Then $d'(\lambda) = 1$ and, for all $v \in \mathbb{N}^*$,

$$\begin{aligned}
\sum_{i\in\mathbb{N}} d'(vi)\gamma^{s'}(vi) &= \sum_{i\in\mathbb{N}} d(vi)\gamma^{1-s'}(vi)\gamma^{s'}(vi) \\
&= \sum_{i\in\mathbb{N}} d(vi)\gamma(vi) \\
&= d(v)\gamma(v) \\
&= d'(v)\gamma^{1-s'}(v)\gamma^{s'}(v),
\end{aligned}$$

as required, where the penultimate equality holds since d is a martingale.

If $d(X \upharpoonright n) \geq \gamma^{s-1}(C_{X\upharpoonright n})$, then $d'(X \upharpoonright n) \geq \gamma^{s-1}(C_{X\upharpoonright n})\gamma^{1-s'}(C_{X\upharpoonright n}) = \gamma^{s-s'}(C_{X\upharpoonright n})$.

Since $s - s' < 0$, $\lim_{n\to\infty} \gamma^{s-s'}(X \upharpoonright n) = \infty$. Thus, d' succeeds on X. ◀

Proof of Lemma 23. We know that

$$\mu(v) = \frac{1}{q_n(q_n + q_{n-1})} \text{ and } \mu(v1) = \frac{1}{(q_n + q_{n-1})(2q_n + q_{n-1})},$$

since $q_{n+1} = q_n + q_{n-1}$. It follows that

$$\mu(v1|v) = \frac{q_n}{2q_n + q_{n-1}} < \frac{1}{2}.$$

Moreover,

$$\mu(v1|v) = \frac{q_n}{(2v_n + 1)q_{n-1} + 2q_{n-2}} > \frac{1}{(2v_n + 3)},$$

since $q_{n-2} < q_{n-1} < q_n$. The result follows from Lemma 2. ◀

The following lemma states that it is possible to construct martingales which never go to 0 on any string.

▶ **Lemma 25.** *Let $d : S \to [0, \infty)$ be a martingale (or supermartingale), where S is either Σ^* or $\mathbb{N}^*$. Let $c \in \mathbb{N}$. Then there is a martingale (respectively, supermartingale) $h : S \to [0, \infty)$ such that $h(w) \geq 2^{-c}$ for every $w \in S$, where $S^\infty[h] \supseteq S^\infty[d]$ and $S^\infty_{str}[h] \supseteq S^\infty_{str}[d]$. If d is lower semicomputable (or computable), then h is lower semicomputable (respectively, computable).*

Proof. First, let $S = \Sigma^*$. For any $w \in \Sigma^*$, let $h(w) = d(w) + 2^{-c}$. Then $h(w0) + h(w1)$ is $d(w0) + d(w1) + 2^{-c+1}$. If d is a martingale, then this is $2d(w) + 2^{-c+1}$, which is $2h(w)$. Thus h is a martingale. If d is a supermartingale, the above quantity is upper bounded by $2d(w) + 2^{-c+1}$, hence upper bounded by $2h(w)$. Thus h is a supermartingale. Since $h \geq d$, it follows that $S^\infty[h] \supseteq S^\infty[d]$ and $S^\infty_{\text{str}}[h] \supseteq S^\infty_{\text{str}}[d]$. Also, since h is obtained by the addition of a rational to d, it follows that if d is lower semicomputable (or, computable), then h is lower semicomputable (respectively, computable).

The proof for continued fraction martingales is similar. ◀

▶ **Lemma 26.** *Let $d : \Sigma^* \to [0, \infty)$ be a martingale, where there is a $c \in \mathbb{N}$ such that for all $w \in \Sigma^*$, we have $d(w) \geq 2^{-c}$. Then the function $h : \Sigma^* \to [0, \infty)$ defined by $h = \log_2(d) + c + 1$ is a supermartingale, with $S^\infty[h] \supseteq S^\infty[d]$ and $S^\infty_{str}[h] \supseteq S^\infty_{str}[d]$. If d is lower semicomputable (or computable), then h is lower semicomputable (respectively, computable).*

Proof. Let d and h be as given. Then $0 < h(\lambda) = \log_2(d(\lambda)) + c + 1 < \infty$, since $2^{-c} < d(\lambda) < \infty$. For every $w \in \Sigma^*$, $h(w) = \log_2(d(w)) + c + 1 > 0$. Further, we have

$$\begin{aligned} \frac{h(w0) + h(w1)}{2} &= \frac{\log_2 d(w0) + \log_2 d(w1) + 2c + 2}{2} \\ &\leq \log_2 \left[\frac{d(w0) + d(w1)}{2} \right] + c + 1 \\ &= \log_2 d(w) + c + 1 = h(w), \end{aligned}$$

by Jensen's inequality. Hence h is a supermartingale. Since $d \geq 2^{-c}$, h is a computable real-valued function of d. Hence if d is lower semicomputable (or computable), then h is lower semicomputable (respectively computable). ◀

▶ **Lemma 27.** *Let (a, b) be a subinterval of $[0, 1]$ with rational endpoints, and $W \subseteq \Sigma^*$ be defined by*

$$W = \{w \in \Sigma^* \mid C_w \subseteq (a, b), \nexists u \sqsubset w \; u \in V\}.$$

Then $\sum_{w \in W} \mu(w) = b - a = \mu((a, b))$.

▶ **Lemma 28.** *Let $\langle n_i \rangle_{i \in \mathbb{N}}$ be a monotone non-decreasing sequence of positive integers such that $\sum_{i \in \mathbb{N}} 2^{-n_i} < \frac{1}{2^N} < \infty$. Then $\sum_{i \in \mathbb{N}} \frac{n_i}{2^{-n_i}}$ is upper-bounded by a term computable solely from N and which tends to 0 as $N \to \infty$.*

Proof. For every $k \in \mathbb{N}$, let $f_k = 2^{-n_k}$. Let $g_1 = 0$ and for $k \geq 2$, let $g_k = \sum_{j=1}^{k-1} n_j$. For any sequence $\langle x_k \rangle_{k \in \mathbb{N}}$ of reals, let the forward difference operator Δ be defined by $\Delta x_k = x_{k+1} - x_k$, $k \in \mathbb{N}$. Then, we have $\Delta f_k = 2^{-n_{k+1}} - 2^{-n_k}$ and $\Delta g_k = n_k$. Using summation by parts [7], we know that for any $m \in \mathbb{N}$,

$$\sum_{k=1}^{m} f_k \Delta g_k = f_m g_{m+1} - f_1 g_1 - \sum_{k=1}^{m-1} g_k \Delta f_k.$$

Then, we have,

$$\begin{aligned} \sum_{k=1}^{m} \frac{n_k}{2^{n_k}} &= \sum_{k=1}^{m} f_k \Delta g_k \\ &= \frac{g_{m+1}}{2^{n_m}} - 0 - \sum_{k=1}^{m-1} \sum_{j=1}^{k-1} n_j \left[\frac{1}{2^{n_{k+1}}} - \frac{1}{2^{n_k}} \right]. \end{aligned}$$

The last summation term is negative, so the expression is a sum of positive terms. Moreover, since $g_{m+1} = O(n_m^2)$, the first term tends to 0 as $m \to \infty$. Taking the limit of the entire expression with respect to m, we get[2]

$$\lim_{m \to \infty} \sum_{k=1}^{m-1} \sum_{j=1}^{k-1} n_j \left[\frac{1}{2^{n_k}} - \frac{1}{2^{n_{k-1}}} \right].$$

[2] The limit at this point exists only in $[0, \infty]$ and hence may be ∞.

If $n_k = n_{k-1}$, then the term $2^{-n_k} - 2^{-n_{k-1}}$ is 0, hence the expression on the right is a positive sum involving terms from a a strictly monotone decreasing subsequence $\langle n_{k_i} \rangle_{i \in \mathbb{N}}$, where the largest term is necessarily less than or equal to $n/2^n$. Hence the expression on the right is at most

$$\sum_{k=n}^{\infty} \frac{O((k+1)^2)}{2^k} \leq \sum_{k=n}^{\infty} \frac{o(2^{k/2})}{2^k} = \sum_{k=n}^{\infty} \frac{1}{2^{\omega k/2}} \leq \frac{1}{\sqrt{2}+1} \frac{1}{2^{\frac{n-1}{2}}}, \tag{6}$$

which is a term computable in n and which monotone decreases to 0 as $n \to \infty$. ◀