Directed Planar Reachability is in Unambiguous Log-space

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Abstract

We show that the st-connectivity problem for directed planar graphs can be decided in unambiguous logarithmic space.

1. Introduction

Graph reachability problems are fundamental to complexity theory. The general st-connectivity problem for directed graphs perfectly captures the power of nondeterminism in the context of logarithmic space since it is complete for NL. Various restricted versions of this problem characterize other low-level complexity classes such as L, AC^0, and NC^1 [Ete97, BLMS98, Bar89]. Moreover, though the st-connectivity problem for undirected graphs was recently resolved by Reingold [Rei05], the precise complexity for directed graphs remains less well understood.

A natural and important restriction of the st-connectivity problem is when the graphs involved are planar, which we denote by PLANARREACH in this paper. The complexity of this problem is not yet settled satisfactorily. The best known upper bound in terms of space complexity is NL. Though it is hard for L [Ete97], it is not known whether it is complete for NL. Recently there has been progress in understanding the complexity of PLANARREACH. In particular, Allender and Mahajan [AM04] have demonstrated an algorithm that determines whether or not a given graph is planar and in the case that it is, produces a planar embedding. From the work of Reingold [Rei05] it follows that this can be done in deterministic log-space. Further, Allender, Datta and Roy [ADR05] show that PLANARREACH log-space reduces to the reachability problem for a strict subclass of planar graphs called *grid graphs*. We denote the reachability problem for grid graphs as GGR. From this result and the fact that GGR reduces to its complement [BLMS98], it follows that PLANARREACH reduces to its complement problem of unreachability in planar graphs. Allender et al. [ADR05] also give a direct log-space reduction from PLANARREACH to its complement.

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In this paper we make further progress in understanding the space complexity of PLANARREACH. Building on earlier work, we give a simple argument to show the following upper bound.

Theorem 1. PLANARREACH $\in \mathsf{UL} \cap \mathsf{coUL}$.

Here UL denotes the unambiguous subclass of NL first defined and studied in [BJLR91, AJ93]. A language L is in UL if and only if there exists a nondeterministic log-space machine M accepting L such that, for every instance x, M has at most one accepting computation on input x. Thus, planar reachability can be decided by a nondeterministic machine in log-space with at most one accepting computation.

The class UL and related low-space unambiguous classes have been of interest to researchers [BJLR91, AJ93, Lan97, AL98, RA00, ADR05]. Arguably the most interesting result regarding UL, due to Reinhardt and Allender, is that the nonuniform version of UL contains the whole of NL; that is $\mathsf{NL} \subseteq \mathsf{UL/poly}$ [RA00]. In addition, Allender, Reinhardt, and Zhou showed that, under the hardness assumption that deterministic linear space has functions that can not be computed by circuits of size $2^{\epsilon n}$, the constructions given in [RA00] can be *derandomized* to show that $\mathsf{NL} = \mathsf{UL}$ [ARZ99]. These results give strong indication that NL equals UL. Our result gives further evidence that this equality might hold.

Since PLANARREACH reduces to reachability in grid graphs, for our upper bound it suffices to consider grid graphs. Grid graphs are graphs with vertices located on the planar grid and edges connecting a vertex only with its immediate vertical/horizontal neighbors. Reachability in grid graphs is interesting from a complexity-theoretic point of view. Barrington, Lu, Miltersen, and Skyum showed that st-connectivity on such graphs with constant width captures the complexity of the AC^0 hierarchy [BLMS98]. Recently, the complexity of various restrictions of GGR have been studied in [ADR05, ABC$^+$06]. Specifically, in [ADR05] the authors show that the *layered* grid graph reachability problem is in UL. A layered grid graph is a restriction to only three cardinal directions. It was open whether this upper bound can be extended to the general grid graph reachability. We settle this problem in this paper.

2. Preliminaries

We assume familiarity with the basics of complexity theory and in particular log-space bounded complexity classes NL and its unambiguous version UL. It is well known that checking for st-connectivity for general directed graphs is NL-complete. We consider the st-connectivity problem for planar graphs and grid graphs.

A $n \times n$ *grid graph* is a directed graph whose vertices are $\{0, \ldots, n-1\} \times \{0, \ldots, n-1\}$ so that if $((i_1, j_1), (i_2, j_2))$ is an edge then $|i_1 - i_2| + |j_1 - j_2| = 1$. Grid graphs are a very natural subclass of planar graphs with vertices identified with the $n \times n$ grid on the x-y plane oriented at the origin with directed edges connecting only the immediate vertical and horizontal neighbors. It is convenient to view the edges according to the cardinal directions. For a vertex (i, j), the edge $(i, j) \rightarrow (i, j+1)$ is a north edge, $(i, j) \rightarrow (i, j-1)$ is a south edge, $(i, j) \rightarrow (i+1, j)$ is an east edge, and $(i, j) \rightarrow (i-1, j)$ is a west edge.

The grid graph reachability problem is as follows. Given a grid graph G and vertices s and t, determine if there exists a directed path from s to t in G. The directed planar reachability problem denoted as PLANARREACH is the following: Given a planar graph G and vertices s and t, determine if there exists a directed path from s to t in G. We will use the following two results.

Theorem 2 ([ADR05])**.** PLANARREACH *log-space many-one reduces to* GGR.

Theorem 3 ([BLMS98, ADR05])**.** PLANARREACH *log-space many-one reduces to its complement.*

Because of the above reductions and the fact that UL is closed under log-space many-one reductions, it is enough to show that GGR $\in$ UL. We will focus on this task for the remainder of the paper.

In [RA00] Reinhardt and Allender give a general technique for showing membership in UL which we will make use of.

A *min-unique* graph is a directed graph with positive weights associated with each edge where for every pair of vertices u, v, if there is a path from u to v, then there is a unique minimum weight path from u to v. Here, the weight of a path is the sum of the weights on its edges. (We note that Reinhardt and Allender actually define min-uniqueness for unweighted graphs, but these two definitions are essentially same in our setting where the weights are log-space computable, since we can replace a directed edge with positive weight w with a directed path of length w.)

The following theorem is implicitly given in [RA00]. Its proof uses a clever extension of the inductive counting techniques of Immerman and Szelepcsényi [Imm88, Sze88].

Theorem 4 ([RA00])**.** *Let $\mathcal{G}$ be a class of graphs. If there is a log-space computable function f that on input $G \in \mathcal{G}$ outputs a weighted graph $f(G)$ so that*

(a) $f(G)$ is min-unique and

(b) G has an $s-t$ path if and only if $f(G)$ has an $s-t$ path,

then the st-connectivity problem for $\mathcal{G}$ is in UL.

3. Planar Reachability is in UL $\cap$ coUL

We now prove Theorem 1. In light of Theorems 2, 3, and 4, it suffices to show a log-space computable positive weight function which produces a min-unique graph for grid graphs. (In fact Theorem 3 is not necessary, as the algorithm of Reinhardt and Allender actually places reachability for min-unique graphs in UL $\cap$ coUL).

Proof of the Theorem 1. Let G be a grid graph with the rows and columns of G indexed from 0 to $n-1$. We define a weight function w on the edges of G as follows.

$$w(e) = \begin{cases} n^4 & \text{if } e \text{ is an east or west edge} \\ i + n^4 & \text{if } e \text{ is a north edge in column } i \\ -i + n^4 & \text{if } e \text{ is a south edge in column } i \end{cases}$$

Clearly, w is log-space computable. Since $i < n$, the weight on any edge is positive. Since a minimum weight path has to be simple, we will only focus on simple paths. The weight of any path P in G, denoted by $w(P)$ is of the form $a + bn^4$. It is easy to see that $|a|, b < n^3$ (a could be negative). For a given path P, let $a(P)$ denote the 'a' component and $b(P)$ denote the 'b' component of its weight. Here, $a(P)$ serves to weight a path's vertical edges while $b(P)$ serves to count the total number of edges in the path.

Let P_1 and P_2 be two paths in G having the same weight. Then it is easy to see that $a(P_1) = a(P_2)$ and $b(P_1) = b(P_2)$. To see this, let $w(P_1) = a_1 + b_1 n^4$ and $w(P_2) = a_2 + b_2 n^4$. Since,

$$\begin{aligned} & w(P_1) &=& \; w(P_2) \\ \Rightarrow \; & (a_1 - a_2) + (b_1 - b_2)n^4 &=& \; 0 \\ \Rightarrow \; & a_1 = a_2 & \text{and} & \; b_1 = b_2 \end{aligned}$$

The final implication follows since $|a_i|$'s and b_i, and hence their respective differences, are bounded by n^4. Now we will argue that, with respect to this weight function for any u and v, the minimum weight path from u to v, if it exists, is unique.

First we prove a very nice property of this weight function that the 'a' component of the weight of any nontrivial simple cycle in G is non-zero. In fact, we prove the following stronger property of this weight function. For a simple cycle C, let $A(C)$ denote the number of unit squares it encloses.

Lemma 5. *Let C be a simple directed cycle in G. Then $a(C) = +A(C)$ if C is a counter-clockwise cycle and $a(C) = -A(C)$ if C is a clockwise cycle.*

Proof. It suffices to prove the lemma for a counter-clockwise simple cycle. This is because, for a clockwise cycle C, $a(C) = -a(-C)$ where $-C$ is the counter-clockwise cycle obtained by reversing the edges in C.

Let C be a counter-clockwise cycle in G. Look at the restriction of the cycle to the set of edges between two consecutive rows, say j and $j+1$. We can view this as an ordered set, where an edge e appears before an edge e' in the ordering if e is to the left of e' in the graph. Denote this ordered set by S_j.

If we order these edges from left to right it alternates between south and north edges with the left most edge being a south edge and the rightmost edge being a north edge. Suppose not, then there are two adjacent north edges in S_j, say $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$. This implies that C does not have any edge between e_1 and e_2 in G. Also there is a simple path from v_2 to u_1 and from v_1 to u_2 without taking any edge between e_1 and e_2 in G. This implies that the paths intersect and hence contradicts the fact that C is a simple cycle. The reason why the first edge is a south edge is because it is a counter-clockwise cycle.

Now lets look at the set of squares that lie between row j and $j+1$. Denote this set by R_j. $a(C)$ restricted to S_j counts the number of squares in R_j that lie between adjacent south and north edges, with the north edge being to the right of the south edge (cf. Figure 1). This is because the weight of the kth south edge plus the weight of the kth north edge is equal to the number of squares between these two edges.

A square in R_j is in C iff it is between a south and a north edge. This is because if we look at a partition of the set R_j induced by the edges in S_j, then the partitions alternately fall within and outside of C, with the set of squares between the first south and north edge lying within the cycle.

Now we sum our index j from 0 to $n-2$ and thus get that the sum of the edge weights of the cycle is equal to the number of squares it encloses. □

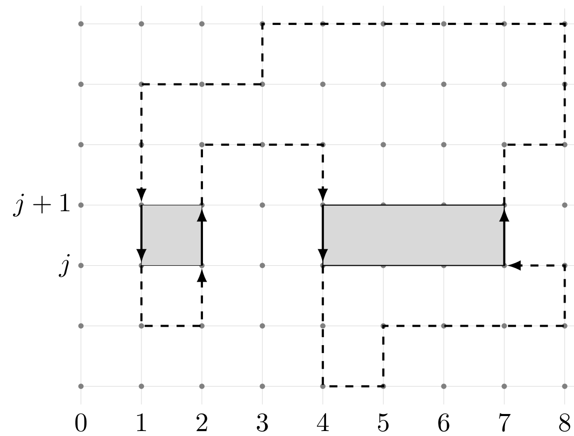

Figure 1. A view of a grid graph between row j and $j+1$ of a counter-clockwise cycle. The cycle has an 'a' component weight of $(2-1)+(7-4)=4$ with respect to rows j and $j+1$, equal to the number of unit squares it encloses.

Lemma 6. *Let G be a grid graph. With respect to the weight function w, for any two vertices u and v, the minimum weight path from u to v, if one exists, is unique.*

Proof. Suppose there exist two different minimum weight paths P_1 and P_2 between u and v. Let u' be the vertex at which P_1 and P_2 diverge for the first time and let v' be the vertex where they meet after their first divergence. Denote the subpath of P_1 from u' to v' by P_1' and the subpath of P_2 from u' to v' by P_2' (cf. Figure 2). u' exists since P_1 and P_2 are not the same, and existence of u' implies existence of v'.

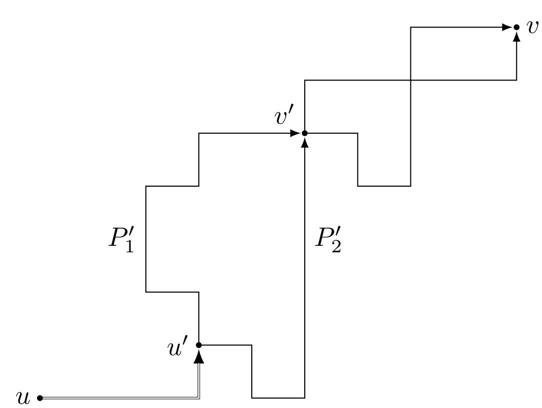

Figure 2. Paths P_1, P_2 from u to v.

If P_1' and P_2' have different weights then without loss of generality assume $w(P_1') < w(P_2')$. This implies that the subpath of P_2 from v' to v has lesser weight than the subpath of P_1 from v' to v. Hence taking P_1 from u to v' and then taking P_2 to v gets a path of lesser weight from u to v, hence a contradiction.

On the other hand, suppose P'_1 and P'_2 have the same weight. Then $a(P'_1) = a(P'_2)$. Now consider a simple cycle C from u' to v' following the path P'_1 and back to u' following the path $-P'_2$. Here for a path P, $-P$ denotes the path obtained by reversing the edges in P. It is clear that $a(-P) = -a(P)$ for any path P. Hence $a(C) = a(P'_1) - a(P'_2) = 0$. This is a contradiction since C is a nontrivial simple cycle and hence $|a(C)| > 0$ by Lemma 5. □

Allender, Datta and Roy show that, if given an embedding on the torus of a graph of genus 1, the st-connectivity problem is reducible (in deterministic log-space) to the planar case [ADR05]. As a consequence of Theorem 1, we get the following.

Corollary 7. *The directed st-connectivity problem for graphs of genus 1 is in* $\mathsf{UL} \cap \mathsf{coUL}$ *(when given an embedding).*

4. Conclusion

We have shown that the st-connectivity problem for directed planar graphs can be decided in $\mathsf{UL} \cap \mathsf{coUL}$, improving over the known upper bound of NL. Many open questions remain. For instance, can we show that PLANARREACH $\in \mathsf{L}$? Or can we even show any deterministic space upper bound for PLANARREACH better than the well-known $\mathcal{O}(\log^2 n)$ upper bound given by Savitch's theorem? Can we show that the general st-connectivity problem can be done unambiguously?

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