Lecture 7: Positive Semidefinite Matrices

Rajat Mittal

IIT Kanpur

The main aim of this lecture note is to prepare your background for semidefinite programming. We have already seen some linear algebra. Now, we will see the concept of eigenvalues and eigenvectors, spectral decomposition and special classes of matrices. The class of positive semidefinite matrices will be of special interest to us. We will look at the properties of positive semidefinite matrices and the cone formed by them.

Remember, matrices are linear operators and every linear operator can be represented by a matrix (if we fix the basis). There were two important theorems that we covered earlier.

Theorem 1. *Let M be an $m \times n$ matrix. The dimension of the image of M is known as the* rank *of M ($rank(M)$) and the dimension of kernel of M is known as the nullity of M ($nullity(M)$). By the famous rank-nullity theorem,*

$$rank(M) + nullity(M) = n.$$

Theorem 2. *The column rank of a matrix M is same as the row rank of M.*

1 Eigenvalues and eigenvectors

Consider two vector spaces V and W over real numbers. A matrix $M \in L(V, W)$ is square if $dim(V) = dim(W)$. In particular, a matrix $M \in L(V)$ is always square.

Consider a matrix $M \in L(V)$, any vector $v \in V$ satisfying,

$$Mv = \lambda v \text{ for some } \lambda \in \mathbb{R},$$

is called the *eigenvector* of matrix M with *eigenvalue* λ.

Exercise 1. Given two eigenvectors v, w, when is their linear combination an eigenvector itself?

The previous exercise can be used to show that all the eigenvectors corresponding to a particular eigenvalue form a subspace. This subspace is called the *eigenspace* of the corresponding eigenvalue.

An eigenvalue λ of an $n \times n$ matrix M satisfies the equation

$$det(\lambda I - M) = 0,$$

where $det(M)$ denotes the determinant of the matrix M. The polynomial $det(\lambda I - M) = 0$, in λ, is called the *characteristic polynomial* of M. The characteristic polynomial has degree n and will have n roots in the field of complex numbers. Though, these roots might not be real.

It can be shown that if λ is a root of characteristic polynomial then there exist at least one eigenvector corresponding to λ. We leave it as an exercise.

Exercise 2. Give an example of a matrix with no real roots of the characteristic polynomial.

The next theorem says that eigenvalues are preserved under basis transformation.

Theorem 3. *Given a matrix P of full rank, matrix M and matrix $P^{-1}MP$ have the same set of eigenvalues.*

Proof. Suppose λ is an eigenvalue of $P^{-1}MP$, we need to show that it is an eigenvalue for M too. Say λ is an eigenvalue with eigenvector v. Then,

$$P^{-1}MPv = \lambda v \Rightarrow M(Pv) = \lambda Pv.$$

Hence Pv is an eigenvector with eigenvalue λ.

The opposite direction follows similarly. Given an eigenvector v of M, it can be shown that $P^{-1}v$ is an eigenvector of $P^{-1}MP$.

$$P^{-1}MP(P^{-1}v) = P^{-1}Mv = \lambda P^{-1}v$$

Hence proved. □

Exercise 3. Where did we use the fact that P is a full rank matrix?

2 Spectral decomposition

Exercise 4. Let v_1, v_2 be two eigenvectors of a matrix M with distinct eigenvalues. Show that these two eigenvectors are linearly independent.

This exercise also shows: sum of the dimensions of eigenspaces of an $n \times n$ matrix M can't exceed n.

Given an $n \times n$ matrix M, it need not have n linearly independent eigenvectors. The matrix M is called *diagonalizable* iff the set of eigenvectors of M span the complete space $\mathbb{R}^n$.

For a diagonalizable matrix, the basis of eigenvectors need not be an orthogonal basis. We will be interested in matrices which have an orthonormal basis of eigenvectors.

Suppose a matrix M has an orthonormal basis of eigenvectors. Let $\lambda_1, \lambda_2, \cdots, \lambda_n \in \mathbb{R}$ be the n eigenvalues with the corresponding eigenvectors $u_1, u_2, \cdots, u_n$. Define D to be the diagonal matrix with $D_{ii} = \lambda_i$ for every i. Let U be the matrix with i-th column being u_i. Since u_i's are orthonormal, $U^T = U^{-1}$.

Exercise 5. Show that $M = UDU^T$. Remember, to show that two matrices are same, we only need to show that their action on a basis is same.

So, if a matrix M has an orthonormal set of eigenvectors, then it can be written as UDU^T. This implies that $M = M^T$. We call such matrices *symmetric*.

What about the reverse direction? *Spectral decomposition* shows that every symmetric matrix has an orthonormal set of eigenvectors. Before proving spectral decomposition, let us look at the eigenvalues and eigenvectors of a symmetric matrix.

Lemma 1. *Let $Mu = \lambda_1 u$ and $Mw = \lambda_2 w$, where λ_1 and λ_2 are not equal. Then*

$$u^T w = 0.$$

Proof. Notice that $\lambda_1, \lambda_2, u, w$ need not be real. From the conditions given in the lemma,

$$\lambda_1 u^T w = (Mu)^T w = u^T(Mw) = \lambda_2 u^T w.$$

Where did we use the fact that M is symmetric?

Since $\lambda_1 \neq \lambda_2$, we get that $u^T w = 0$. □

Let M be a symmetric matrix. A priori, it is not even clear that all the roots of $det(M - \lambda I)$ are real. Let us first prove that all roots of the characteristic polynomial are real.

Lemma 2. *Given an $n \times n$ symmetric matrix M, all roots of $det(\lambda I - M)$ are real.*

Proof. Let λ be a root of $det(\lambda I - M)$. Suppose it is not real,

$$\lambda = a + ib, \text{ where } b \neq 0.$$

Since $\lambda I - M$ is zero, the kernel of $\lambda I - M$ is not empty. Hence, there exists a vector $v = x + iy$, such that

$$M(x + iy) = (a + ib)(x + iy).$$

Taking the adjoint of this equation and noting $M^* = M$ (M is real and symmetric),

$$M(x - iy) = (a - ib)(x - iy).$$

Using Lem. 1, we know that $x + iy$ and $x - iy$ should be orthogonal to each other.

Exercise 6. Prove that $x + iy$ and $x - iy$ can not be orthogonal to each other by taking their inner product.

□

Now we are ready to prove spectral decomposition.

Theorem 4. *Spectral decomposition: For a symmetric matrix $M \in \mathbb{R}^{n\times n}$, there exists an orthonormal basis $x_1, \cdots, x_n$ of $\mathbb{R}^n$, s.t.,*

$$M = \sum_{i=1}^{n} \lambda_i x_i x_i^T.$$

Here, $\lambda_i \in \mathbb{R}$ for all i.

Note 1. It means that any symmetric matrix $M = U^T DU$. Here D is the diagonal matrix with eigenvalues and U is the matrix with columns as eigenvectors.

Exercise 7. Show that x_i is an eigenvector of M with eigenvalue λ_i.

Note 2. $u^T w$ is a scalar, but uw^T is a matrix.

Note 3. The λ_i's need not be different. If we collect all the x_i's corresponding to a particular eigenvalue λ, the space spanned by those x_i's is the eigenspace of λ.

Proof of Thm. 4. Proof of spectral theorem essentially hinges on the following lemma.

Lemma 3. *Given an eigenspace S (of eigenvalue λ) of a symmetric matrix M, the matrix M acts on the space S and $S^\perp$ separately. In other words, $Mv \in S$ if $v \in S$ and $Mv \in S^\perp$ if $v \in S^\perp$.*

Proof of lemma. Since S is an eigenspace, $Mv \in S$ if $v \in S$. This shows that M^T preserves the subspace S.
Suppose $v_1 \in S^\perp$, $v_2 \in S$, then $Mv_2 = M^T v_2 \in S$. So,

$$0 = v_1^T M^T v_2 = (Mv_1)^T (v_2).$$

This shows that $Mv_1 \in S^\perp$. Hence, matrix M acts separately on S and $S^\perp$. □

We have already shown that the eigenvectors of a symmetric matrix corresponding to different eigenvalues are orthogonal (Lem. 1). Also, it was shown that every root of the characteristic polynomial is real, so there are n real roots (Lem. 2). Though some roots might be present with multiplicities more than 1.

Assume that we list out all possible eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$ with their eigenspaces $P_1, P_2, \cdots, P_k$. If $\sum_{i=1}^{k} dim(P_i) = n$, then we are done. If not say the remaining space is P_{k+1}.

Since eigenvalues do not change under a basis transformation (Thm. 3), we can look at M in the bases of $P_1, P_2, \cdots, P_{k+1}$. Lem. 3 implies that matrix M looks like,

$$\begin{pmatrix} \lambda_1 & & & & & & \\ & \lambda_1 & & & & & \\ & & \ddots & & & & \\ & & & \lambda_k & & & \\ & & & & \ddots & & \\ & & & & & \lambda_k & \\ & & & & & & C \end{pmatrix}.$$

We can assume that C is non-zero (why?). Then fundamental theorem of algebra says that $det(\lambda I - C)$ has a root. Since that will also be a root of $det(\lambda I - M)$, it has to be real. But then this real root will have at least one eigenvector. This is a contradiction, since we had listed all possible eigenvalues and their eigenspaces.

Exercise 8. Given the spectral decomposition of M, what is the spectral decomposition of M^T?

□

It was shown before that any matrix with orthonormal set of eigenvectors is a symmetric matrix. Hence, spectral decomposition provides another characterization of symmetric matrices.

Clearly the spectral decomposition is not unique (essentially because of the multiplicity of eigenvalues). But the eigenspaces corresponding to each eigenvalue are fixed. So there is a unique decomposition in terms of eigenspaces and then any orthonormal basis of these eigenspaces can be chosen.

2.1 Orthogonal matrices

A matrix M is orthogonal if $MM^T = M^TM = I$. In other words, the columns of M form an orthonormal basis of the whole space. Orthogonal matrices need not be symmetric, so roots of their characteristic polynomial need not be real. For an orthogonal matrix, $M^{-1} = M^T$.

Exercise 9. Give an example of an orthogonal matrix which is not symmetric.

Orthogonal matrices can be viewed as matrices which implement a change of basis. Hence, they preserve the angle (inner product) between the vectors. So for an orthogonal M,

$$u^Tv = (Mu)^T(Mv).$$

Exercise 10. Prove the above equation.

If two matrices A, B are related by $A = M^{-1}BM$, where M is orthogonal, then they are called *orthogonally similar*. If two matrices are orthogonally similar then they are similar.

Spectral theorem can be stated as the fact that symmetric matrices are orthogonally similar to a diagonal matrix. In this case, the diagonal of the diagonal matrix contains the eigenvalues of the symmetric matrix.

Exercise 11. What is the rank of an orthogonal matrix?

3 Extra reading: singular value decomposition

Singular value decomposition is one of the most important factorizations of a matrix. The statement says,

Theorem 5. *Given a linear operator M in $L(V, W)$. There exists a decomposition of the form:*

$$M = \sum_{i=1}^{r} s_i y_i x_i^T$$

Where $x_1, \cdots, x_r$ (called right singular vectors) and $y_1, \cdots, y_r$ (called left singular vectors) are orthonormal basis of V and W respectively. The numbers $s_1, \cdots, s_r$ (called singular values) are positive real numbers and r itself is the rank of the matrix M.

The statement of the theorem can also be written as $M = A\Delta B^T$, where $A \in L(W), B \in L(V)$ are orthogonal matrices and Δ is the diagonal matrix of singular values. With this interpretation, any linear operation can be viewed as rotation in subspace V then scaling the standard basis and then another rotation in W subspace.

The statement of singular value decomposition is easy to prove if we don't need any condition on y_i's. Any basis of V will be sufficient to construct such a decomposition (why?). We can even choose all singular values to be 1 in that case. But it turns out that with the singular values we can make the y_i's to be orthonormal.

The proof of singular value decomposition follows by applying spectral decomposition on matrices MM^T and M^TM. Note that if v is an eigenvector of MM^T then Mv is an eigenvector of M^TM. Hence, M^TM and MM^T have the same set of eigenvalues.

Suppose the eigenvectors of M^TM are x_i's for eigenvalues λ_i, then eigenvectors of MM^T are $\frac{Mx_i}{\|Mx_i\|} = y_i$'s.

Exercise 12. Prove the above statement.

If we reduce A to its row echelon form, it can be seen that $rank(A) = rank(A^T A)$. Hence, it is enough to specify the action of M on x_i's. So,

$$M = \sum_{i=1}^{r} \|Mx_i\| y_i x_i^T$$

Exercise 13. Prove that $\|Mx_i\| = \sqrt{\lambda_i}$.

This implies the singular value decomposition.

$$M = \sum_{i=1}^{r} \sqrt{\lambda_i} y_i x_i^T = \sum_{i=1}^{r} s_i y_i x_i.$$

The eigenvectors of MM^T are left singular vectors and eigenvectors of $M^T M$ are right singular vectors of M. The eigenvalues of MM^T or $M^T M$ are the squares of the singular values of M.

4 Positive semidefinite matrices

A matrix M is called positive semidefinite if it is symmetric and all its eigenvalues are non-negative. If all eigenvalues are strictly positive then it is called a positive definite matrix.

In many references, you might find another definition of positive semidefiniteness. A matrix $M \in L(V)$ will be called positive semidefinite iff,

1. M is symmetric,
2. $v^T M v \geq 0$ for all $v \in V$.

If the matrix is symmetric and

$$v^T M v > 0, \ \forall v \in V,$$

then it is called positive definite. When the matrix satisfies opposite inequality it is called negative definite.

The two definitions for positive semidefinite matrix turn out be equivalent. In the next theorem, we identify many different definitions of positive semidefinite matrices to be equivalent.

Theorem 6. *For a symmetric $n \times n$ matrix $M \in L(V)$, following are equivalent.*

1. *$v^T M v \geq 0$ for all $v \in V$.*
2. *All the eigenvalues are non-negative.*
3. *There exist a matrix B, s.t., $B^T B = M$.*
4. *Gram matrix of vectors $u_1, \cdots, u_n \in U$, where U is some vector space. Hence*

$$\forall i, j : \ M_{i,j} = v_i^T v_j.$$

Proof. 1 ⇒ 2: Say λ is an eigenvalue of M. Then there exist eigenvector $v \in V$, s.t., $Mv = \lambda v$. So $0 \leq v^T M v = \lambda v^T v$. Since $v^T v$ is positive for all v, implies λ is non-negative.

2 ⇒ 3: Since the matrix M is symmetric, it has a spectral decomposition.

$$M = \sum_i \lambda_i x_i x_i^T$$

Define $y_i = \sqrt{\lambda_i} x_i$. This definition is possible because λ_i's are non-negative. Then,

$$M = \sum_i y_i y_i^T.$$

Define B to be the matrix whose columns are y_i. Then it is clear that $B^TB = M$. From this construction, B's columns are orthogonal. In general, any matrix of the form B^TB is positive semi-definite. The matrix B need not have orthogonal columns (it can even be rectangular).

But this representation is not unique and there always exists a matrix B with orthogonal columns for M, s.t., $B^TB = M$. This decomposition is unique if B is positive semidefinite. The positive semidefinite B, s.t., $B^TB = M$, is called the square root of M.

Exercise 14. Prove that the square root of a matrix is unique.

Hint: Use the spectral decomposition to find one of the square root. Suppose A is any square root of M. Then use the spectral decomposition of A and show the square root is unique (remember the decomposition to eigenspaces is unique) .

$3 \Rightarrow 4$: We are given a matrix B, s.t., $B^TB = M$. Say the rows of B are $u_1, \cdots, u_n$. Then, from the definition of matrix multiplication,

$$\forall i, j: \; M_{i,j} = v_i^T v_j$$

Exercise 15. Show that for a positive semidefinite matrix $M \in L(V)$, there exists $v_1, \cdots, v_n \in V$, s.t, M is a gram matrix of $v_1 \cdots, v_n$.

$4 \Rightarrow 1$: Suppose M is the gram matrix of vectors $u_1, \cdots, u_n$. Then,

$$x^TMx = \sum_{i,j} M_{i,j} x_i x_j = \sum_{i,j} x_i x_j (v_i^T v_j),$$

where x_i is the i^{th} element of vector x. Define $y = \sum_i x_i v_i$, then,

$$0 \geq y^T y = \sum_{i,j} x_i x_j (v_i^T v_j) = x^T M x.$$

Hence $x^TMx \geq 0$ for all x.

Exercise 16. Prove that $2 \Rightarrow 1$ and $3 \Rightarrow 1$ directly.

□

Remark: A matrix M of the form $M = \sum_i x_i x_i^T$ is positive semidefinite (Exercise: Prove it), even if x_i's are not orthogonal to each other.

Remark: A matrix of the form yx^T is a rank one matrix. It is rank one because all columns are scalar multiples of y. Similarly, all rank one matrices can be expressed in this form.

Exercise 17. A rank one matrix yx^T is positive semi-definite iff y is a positive scalar multiple of x.

4.1 Some examples

- An $n \times n$ identity matrix is positive semidefinite. It has rank n. All the eigenvalues are 1 and every vector is an eigenvector. It is the only matrix with all eigenvalues 1 (prove it).
- The all 1's matrix J $(n \times n)$ is a rank one positive semidefinite matrix. It has one eigenvalue n and rest are zero.
- The matrix

  $$M = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

  is positive semidefinite. Because, the quadratic form $x^TMx = (x_1 - x_2)^2$, where x_1, x_2 are two components of x.
- Suppose any symmetric matrix M has maximum eigenvalue λ. The matrix $\lambda' I - M$, where $\lambda' \geq \lambda$ is positive semidefinite.

5 Properties of semidefinite matrices

Positive semidefinite matrices are symmetric matrices whose eigenvalues are non-negative. They can also be thought of as the gram matrix of a set of vectors. Let us look at their special properties and the cone generated by them.

5.1 Principal submatrix

A principal submatrix P of a matrix M is obtained by selecting a subset of rows and the same subset of columns. If M is positive semidefinite then all its principal submatrices are also positive semidefinite.

This follows by considering the quadratic form $x^T Mx$ and looking at the components of x corresponding to the defining subset of principal submatrix. The converse is trivially true.

Exercise 18. Show that the determinant of a positive semidefinite matrix is non-negative. Hence, show that all the principal minors are non-negative. Actually the converse also holds true, i.e., if all the principal minors are non-negative then the matrix is positive semidefinite.

5.2 Diagonal elements

If the matrix is positive semidefinite then its diagonal elements should *dominate* the non-diagonal elements. The quadratic form for M is,

$$x^T Mx = \sum_{i,j} M_{i,j} x_i x_j. \tag{1}$$

Here x_i's are the respective components of x. If M is positive semidefinite then Eqn. 1 should be non-negative for every choice of x.

By choosing x to be a standard basis vector e_i, we get $M_{ii} \geq 0,\ \forall i$. Hence, all diagonal elements are non-negative and $tr(M) \geq 0$. If x is chosen to have only two nonzero entries, let's say at i and j position, then Eqn. 1 implies,

$$M_{i,j} \leq \sqrt{M_{ii} M_{jj}} \leq \frac{M_{ii} + M_{jj}}{2}.$$

Where the second inequality follows from AM-GM inequality. This shows that any off diagonal element is less than the diagonal element in its row or in its column.

5.3 Composition of semidefinite matrices

- The direct sum matrix $A \oplus B$,

  $$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

  is positive semidefinite iff A and B both are positive semidefinite. This can most easily be seen by looking at the quadratic form $x^T (A \oplus B)x$. Divide x into x_1 and x_2 of the required dimensions, then

  $$x^T (A \oplus B)x = x_1^T A x_1 + x_2^T B x_2.$$

- The tensor product $A \otimes B$ is positive semidefinite iff A and B are both positive semidefinite or both are negative semidefinite. This follows from the fact that given the eigenvalues $\lambda_1, \cdots, \lambda_n$ for A and $\mu_1, \cdots, \mu_m$ for B; the eigenvalues of $A \otimes B$ are

  $$\forall i, j: \ \lambda_i \mu_j.$$

- The sum of two positive semidefinite matrices is positive semidefinite.

- The product of two positive semidefinite matrices need not be positive semidefinite.

  Exercise 19. Give an example of two positive semidefinite matrices whose product is not positive semidefinite.

- The Hadamard product of two positive semidefinite matrices A and B, $A \circ B$, is also positive semidefinite. Since A and B are positive semidefinite for some vectors $u_1, \cdots, u_n$ and $v_1, \cdots v_n$. The Hadamard matrix will be the gram matrix of $u_i \otimes v_i$'s. Hence it will be positive semidefinite.
- The inverse of a *positive definite* matrix is positive definite. The eigenvalues of the inverse are inverses of the eigenvalues.
- The matrix $P^T MP$ is positive semidefinite if M is positive semidefinite.

5.4 Schur's complement

Given a 2×2 block matrix,

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

the *Schur complement* of the matrix D in M is $A - BD^{-1}C$. This gives a criteria to decide if a 2×2 symmetric block matrix is positive definite or not.

Theorem 7. *Suppose M is a symmetric 2×2 block matrix,*

$$M = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}.$$

It is positive definite iff D and the Schur complement $A - BD^{-1}B^T$, both are positive definite.

Proof. Notice that,

$$M = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A - BD^{-1}B^T & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix}^T \tag{2}$$

It is known that,

$$\begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} I & -BD^{-1} \\ 0 & I \end{pmatrix}.$$

Hence $M = P^T NP$ where P is invertible and N is a block diagonal matrix. So M is positive definite if and only if N is positive definite. It is easy to check when a block diagonal matrix is positive definite. That exactly gives us that D and the Schur complement, $A - BD^{-1}B^T$, both have to be positive definite. □

Exercise 20. Given a matrix,

$$M = \begin{pmatrix} I & B \\ B & I \end{pmatrix},$$

where B is symmetric, show that it is positive definite iff $I \pm B \succ 0$.

6 Positive semidefinite cone

Consider the vector space of symmetric $n \times n$ matrices, $\mathbb{R}^{\frac{n(n+1)}{2}}$. We focus on the set of positive semidefinite matrices in this space.

It was seen that if M, N are positive semidefinite, then $\alpha M + \beta N$ is also positive semidefinite for positive α, β. Hence, the set of positive semidefinite matrices is a convex cone in $\mathbb{R}^{\frac{n(n+1)}{2}}$. The cone is denoted $\mathcal{S}_n$.

If $M \succeq 0$ then $-M$ is not positive semi-definite. So the cone $\mathcal{S}_n$ does not contain a line. If we look at the positive definite matrices. They form the interior of the cone. To prove this, we show that for any positive definite matrix M, there exist a ball of size ϵ centered at M and contained in the cone of positive semidefinite matrices.

Theorem 8. *If given an $n \times n$ matrix $M \succ 0$ (positive definite). Then, $M - \epsilon N \succeq 0$ for small enough ϵ and all symmetric N whose norm is* 1*.*

Proof. The norm of N is 1, i.e., $N \bullet N = 1$. So, every entry of N is at-most 1 (exercise: why ?). For every unit vector v, every element is bounded by 1 too. So $v^T N v = \sum_{i,j} v_i v_j N_{ij} \leq n^2$.

Exercise 21. The choice $\epsilon = \frac{\lambda_n}{2n^2}$ will work, where λ_n is the least eigenvalue of M.

□

Identity is positive definite, so interior is not empty.

Hence, $\mathcal{S}_n$ is a convex cone that does not contain a line and has non-empty interior. This implies that the cone $\mathcal{S}_n$ is proper. Define the generalized inequality with respect to this cone.

$$M \succeq N \Leftrightarrow M - N \succeq 0$$

The positive semidefinite cone is generated by all rank one matrices xx^T. They form the extreme rays of the cone. The positive definite matrices lie in the interior of the cone. The positive semidefinite matrices with at least one zero eigenvalue are on the boundary.

7 Assignment

Exercise 22. Prove that if λ is a root of the characteristic polynomial, then there exist at least one eigenvector for λ.

Exercise 23. Show that the matrix M and M^T have the same singular values.

Exercise 24. Read about polar decomposition and prove it using singular value decomposition.

Exercise 25. Read about tensor product of two matrices in Wikipedia.

Exercise 26. What are the eigenvalues of $A \otimes B$, where A, B are symmetric matrices and $\otimes$ denotes the tensor product?

Exercise 27. Give a characterization of the linear operators over $V \otimes W$ in terms of linear operators over V and W. Remember that they form a vector space.

Exercise 28. Show that $\langle v|A|w\rangle = \sum_{ij} A_{ij} v_i^* w_j$.

Exercise 29. The multiplicity of a root of characteristic polynomial is known as the *algebraic multiplicity*. The dimension of the eigenspace of that root is known as the *geometric multiplicity*. It is known that geometric multiplicity is less than algebraic multiplicity.

Show that the geometric multiplicity is same as algebraic multiplicity for a symmetric matrix M.