# Lecture 1: Linear Algebra in Theoretical Computer Science

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This lecture is a advertisement for this course. We will look at what we will be covering and an amazing application of linear algebra.

*Exercise 1.* What is linear algebra?

Almost every area of mathematics is filled with applications of linear algebra. To name a few:

- Geometry: to define lines, rotations we use linear equations and matrices.
- Number theory: many structures, algorithms require understanding of vector spaces formed by number theoretic objects.
- Functional analysis: looks at spaces of functions and linear transformations over them.
- Combinatorics: various situations can be modelled by vector spaces and matrices.
- Theoretical computer science: solving linear equations, eigenvalue/eigenvector methods are at the core of many algorithms and theorems.

*Exercise 2.* Can you think of other applications of linear algebra?

Finding linearity in any structure amounts to simplifying the study of that structure. Even for non-linear systems, we try to approximate them using linear systems. These applications have made linear algebra an important part of any science or engineering curriculum.

If we just look at theoretical computer science (TCS), there is a long list of applications in Markov chains, error correcting codes, quantum computing, communication complexity and so on. It will be impossible to cover all these applications in one course. Our target in this course will be modest, we will take two particular areas of TCS which owe their existence to linear algebra and study them. First part of the course will focus on linear and semidefinite programming. Later part of the course will deal with spectral graph theory.

Linear/semidefinite programming are special cases of convex programming. There are various methods to solve these classes of optimization problems. The idea of this course is *not* to look at those solvers, but fouce our attention on the properties of linear/semidefinite programs and their applications. Semidefinite programming can be thought of as a generalization of linear programming.

After finishing the basics of linear algebra, we will learn linear programming, duality and its applications. Then, linear programs will be generalized to semidefinite programs and their applications will be studied.

Another very interesting connection between graph theory and linear algebra is made by looking at the eigenvalues and eigenvectors of associated matrices. This field is known as *spectral graph theory*. Using these eigenvalues, various properties of graphs like cuts, expansion, convergence of random walk and sparsification can be studied. These techniques can even solve linear equations approximately (giving it back to linear algebra).

Just a warning, I have given the rough plan, grading scheme and content of the course. We will mostly stick to it; though, given that it is an elective course, these things can change depending upon your interest, my interest and time. Please look at the course webpage for details about the administrative part of the course.

Let us finish by taking an example in extremal combinatorics which shows the power of linear algebra. This example and its exposition is taken from Babai and Frankl's book titled *Linear algebra methods in combinatorics*.

### 1 An illustration

Assume that you go to the head of your institute to make clubs in the university. Every club gets funding from the university. So your motivation is to make as many clubs as possible, but the head wants to minimize the number of clubs. The head really likes even numbers and puts the following conditions.

- All clubs should have even number of members.

- All pairwise intersection of clubs should be even.

*Note 1.* We assume that no two clubs can have exactly the same members, otherwise you can have infinite clubs.

Assume that there are n students.

*Exercise 3.* Show that you can make at least  $2^{\lfloor n/2 \rfloor}$  clubs.

Since head's strategy is not very successful, one of the mathematicians suggests to the head that even should be replaced by odd in the first rule.

Exercise 4. What if you replace even by odd in both the rules?

Very much against the wishes of the head, rules are changed to,

- All clubs should have odd number of members.
- All pairwise intersection of clubs should be even.

Let us call this rule *size-odd-intersection-even*. Will this reduce the number of possible clubs?

*Exercise 5.* Remember that the number of clubs were exponential in n previously. Can you give a better bound with this *size-odd-intersection-even* rule?

Surprisingly, linear algebra gives a linear bound on number of clubs. We will show now that the you can form at most n clubs under the *size-odd-intersection-even* rule.

The idea is to associate a vector in  $\mathbb{Q}^n$  for every club. It will be shown that these vectors are linearly independent. Hence, there can be at most n clubs.

Exercise 6. Try to find such vectors.

The vectors will be in dimension n, where every co-ordinate corresponds to a student. The vectors are simply the indicator vectors, i.e., the *i*-th entry is 1 if *i*-th student is in the club, otherwise it is 0. For example, if only students 1 and 2 are present in a club, the associated vector is,

$$v_{\{1,2\}} = (1100 \cdots 0)$$

Suppose we have m clubs, there are m vectors associated with them. Call them  $v_1, v_2, \cdots, v_m$ .

*Exercise* 7. What is the inner product of  $v_i$  and  $v_j$ ?

The inner product  $v_i^T v_j$  is the number of members present in both clubs *i* and *j*. The *size-odd-intersection*even rule translates to conditions,

$$v_i^T v_j = \begin{cases} \text{even,} & \text{if } i \neq j \\ \text{odd,} & \text{if } i = j. \end{cases}$$

The final step is to show that these vectors are linearly independent. If not, there exist  $\lambda_1, \lambda_2, \dots, \lambda_m$ , s.t.,

 $\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_m v_m = 0.$ 

Assume that coefficients for only vectors  $i_1, i_2, \dots, i'_m$  are non-zero. Then,

$$\lambda_{i_1}v_{i_1} + \lambda_{i_2}v_{i_2} + \dots + \lambda_{i'_m}v_{i'_m} = 0.$$

*Exercise 8.* Show that, without loss of generality, we can assume  $\lambda_{i_i}$  are integers and not all even.

Again, we can assume that  $\lambda_{i_1}$  is odd. Taking inner product with  $v_{i_1}$ , we get,

 $odd + even + even + \dots + even = 0.$ 

This is a contradiction. Hence, the vectors  $v_1, v_2, \dots, v_m$  are linearly independent. Since they reside in an *n*-dimensional space, m < n.

This proves that there can be at most n clubs.

## 2 Assignment

*Exercise 9.* Read about convex programming.

*Exercise 10.* We showed that under the *size-odd-intersection-even* rule, at most n clubs can be formed with n students. Can you form n clubs (give a lower bound on the number of clubs)?

*Exercise 11.* You are expected to know about *fields.* If you don't, please read about them. Then, prove the independence result above in  $\mathbb{F}_2$ .

*Exercise 12.* What if we replace odd by even and even by odd in the size-odd-intersection-even rule? Can you show a linear upper bound in that case? Can you show an upper bound of exact n? Hint: It will be easier to consider equations in  $\mathbb{F}_2$ .

*Exercise 13.* Define an  $m \times n$  matrix M with rows as vectors  $v_i$ . Define  $A := MM^T$  to be an  $m \times m$  matrix.

- Show for any two matrices P and Q, rank(PQ) < min(rank(P), rank(Q)).
- Show that rank(A) = m.
- Show that  $m \leq n$ .