Lecture 13: Second eigenvalue

Rajat Mittal

IIT Kanpur

We will continue to study the relationship between combinatorial properties of a graph and eigenvalue of its Laplacian. Again, assume that L_G is the Laplacian of G with eigenvalues $\lambda_1 \leq \lambda_2 \cdots \leq \lambda_n$.

We saw the connection between the first eigenvalue and connectedness and then last eigenvalue and coloring properties of graphs in previous lectures. In this note, we will consider the second eigenvalue of Laplacian and show that it *characterizes* the expansion of the graph. This relationship is known as *Cheeger's inequality*.

We will first give one way to represent expansion in a graph. Cheeger's inequality gives both, upper and lower, bounds on this representation. One side of the proof is easy; the proof of other side is via an algorithm called *Fiedler's algorithm*.

For simplicity we will assume that the graph is d-regular. We will suppress subscript in L_G and just call it L, if there is no confusion about the graph.

1 Expansion of a graph

A graph $G = (V, E)$ *expands* well intuitively if for all non-trivial $S \subseteq V$, there are lot of edges between S and $V - S$. Though, since the graph is d-regular, the number of edges can't be higher than $d|S|$ (assume that S is the smaller part).

So, to measure expansion, we need to normalize the number of edges with the total number of edges going out of the subset too. Hence, we can define edge expansion of S as,

$$\phi(S) := \frac{E(S, V - S)}{d|S|}.$$

Here, we only consider non-empty subsets S such that $V - S$ is larger than S, i.e., $1 \leq |S| \leq |V|/2$.

Since d is a constant, and we are only interested in the d-regular case, we will omit the constant factor of d in the denominator.

A graph will have *high* expansion if expansion of every subset is large. So, we define edge expansion of the graph G as,

$$\phi(G) := \min_{S: 1 \leq |S| \leq |V|/2} \phi(S).$$

A graph is called an *expander* if its expansion is large. Constructing expander graphs is a very active area of research in complexity theory. We will look at expanders more closely later in this course.

In this lecture note, we will focus on the relation between the second eigenvalue of the Laplacian of a graph, λ_2, and expansion of the graph.

2 Cheeger's inequality

Let us look at the characterization of λ_2 in terms of the quadratic form.

$$\lambda_2 = \min_{x : x \perp \mathbf{1}} \frac{x^T L x}{x^T x}. \quad (1)$$

Here, $\mathbf{1}$ is the vector with all entries 1. Remember that $\mathbf{1}$ is the eigenvector corresponding to λ_1.

To give an upper bound on λ_2, we just need to pick an x and show that its quadratic form is related to the expansion $\phi(G)$.

A natural choice for x would be the indicator vector for S, where S is the set which minimizes expansion. The indicator vector $\mathbf{1}_S$ is defined to be the vector with 1 at position i if i is in S and 0 otherwise.

Calculating the quadratic form for $x = \mathbf{1}_S$,

$$x^T L x = \sum_{(i,j)\in E} (x_i - x_j)^2 = E(S, V - S).$$

Similarly,

$$x^T x = |S|.$$

Exercise 1. Show that two equations above give the trivial inequality $\lambda_1 \leq \phi(G)$.

So, the quadratic form is related to $\phi(G)$, but x is not perpendicular to $\mathbf{1}$.

Consider the modified vector x (you will show in the assignment that x is just the projection of old x on orthogonal space of $\mathbf{1}$).

$$x_i = \begin{cases} |S| & i \notin S \\ -|V - S| & i \in S \end{cases}$$

The vector x defined above is clearly perpendicular to $\mathbf{1}$. Calculating the quadratic form,

$$x^T L x = \sum_{(i,j)\in E} (x_i - x_j)^2 = E(S, V - S)(|S| - (-|V - S|)^2 = E(S, V - S)|V|^2.$$

For the norm of x,

$$x^T x = |V - S||S|^2 + |S||V - S|^2 = |S||V - S||V|.$$

Using the characterization of λ_2, Eqn. 1,

$$\lambda_2 \leq \frac{E(S, V - S)|V|^2}{|S||V - S||V|}.$$

Noticing that $2|V - S| \geq |V|$,

$$\lambda_2 \leq \frac{2E(S, V - S)}{|S|}.$$

But S was the optimal set for expansion,

$$\frac{1}{2}\lambda_2 \leq \phi(G). \tag{2}$$

We showed a lower bound on $\phi(G)$ using λ_2. You can view the optimization program of λ_2, Eqn. 1, as a relaxation for the optimization problem of expansion.

It is easy to give an upper bound on λ_2, since it is a relaxation of a minimization problem. We did that by constructing a feasible x for the λ_2 optimization problem.

So, we get a lower bound on expansion using λ_2, Eqn. 2. Cheeger's inequality gives an upper bound on $\phi(G)$ using λ_2.

Theorem 1 (Cheeger's inequality). *Given a connected graph G, let L be its Laplacian matrix. Let λ_2 be the second smallest eigenvalue of L and $\phi(G)$ be its expansion. Then,*

$$\frac{\lambda_2}{2} \leq \phi(G) \leq \sqrt{2d\lambda_2}.$$

We have already given the proof of one side. For the other side, to show that λ_2 is a tight bound on expansion, we need to show a rounding algorithm for Eqn. 1. This rounding algorithm is called the *Fiedler's algorithm.*

3 Fiedler's algorithm

The content of this section is largely taken from Luca Trevisan's course notes.

Our task is to round a solution of the λ_2 optimization problem, Eqn. 1. Given a feasible solution of this optimization problem, we need to construct a subset of vertices with small expansion.

Note 1. Giving an upper bound in terms of λ_2 means showing, expansion is small if λ_2 is small.

We are given an x, such that, $x \perp \mathbf{1}$. How should we construct a subset of vertices?

Exercise 2. Can you think of any rounding procedure?

From the lower bound on expansion, we expect to divide the entries of x in two parts, negative and positive. The rounding algorithm is a slight generalization; it chooses a random threshold t and put $i \in S$ if and only if $x_i \geq t$.

To summarize, we can write the complete Fiedler's algorithm.

```
Find the eigenvector v_2 for eigenvalue λ_2.
Pick a random t.
for all i do
    if x_i ≤ t then
        Put i in S
    end
end
Output the cut S, V − S
```

Algorithm 1: Fiedler's algorithm

We will show that there will be a cut $S, V - S$ from this rounding whose expansion is small compared to the objective value with respect to x. Denote by $F(x)$, the objective value of the λ_2 optimization program on vector x.

Lemma 1. *Let $x \perp \mathbf{1}$ and $S_t, V - S_t$ be the cut formed by Fiedler's algorithm using threshold t. If the objective value is $F(x) = \frac{x^T L x}{x^T x}$ then,*

$$\exists t: \ \phi(S_t) \leq \sqrt{2dF(x)} \ \text{and} \ |S_t| \leq |V|/2.$$

Taking x to be the eigenvector for λ_2, we can prove the other side of Cheeger's inequality.

Exercise 3. Show that $\phi(G) \leq \sqrt{2d\lambda_2}$ using Lem. 1.

We only need to prove Lem. 1 now. Ideally, we would like to analyze rounding on vectors orthogonal to $\mathbf{1}$, but it turns out to be difficult.

Instead, we will show that every $x \perp \mathbf{1}$ can be converted into a non-negative vector y with few non-zero entries for the rounding purposes. Then, we will analyze our rounding for such *special vectors*, non-negative vectors with few non-zero entries.

3.1 Conversion to special vectors

The first step will be to convert $x \perp \mathbf{1}$ in to a special vector y.

Lemma 2. *Any vector $x \perp \mathbf{1}$ can be converted into a non-negative vector y with at most $\frac{|V|}{2}$ non-zero entries such that every cut obtained by rounding of y (for different t's) is also present for some t' in the rounding of x. Also,*

$$F(y) \leq F(x).$$

Proof. The method to convert x to y is,

- let m be the median of the coordinates of x, denote $z = x - m\mathbf{1}$.
- split z in to two non-negative vectors with disjoint support, $z = z^+ - z^-$.
- choose y from z^+, z^- with lesser objective value.

Since m is the median, at most half of the coordinates of z are positive and at most half are negative. So, both z^+ as well as z^- have at most $\frac{|V|}{2}$ non-zero entries. You will prove in the assignment that any cut obtained in the rounding of z^+ or z^- is also obtained in the rounding of x.

To prove the lemma, we only need to prove that,

$$\min(F(z^+), F(z^-)) \leq F(x).$$

We will prove it by showing,

$$\min(F(z^+), F(z^-)) \leq F(z) \leq F(x). \tag{3}$$

The part $F(z) \leq F(x)$ is clear because the numerator of objective value does not change from x to z but the denominator increases for z.

Exercise 4. Show that $x^T x \leq z^T z$.

For the other part, let us compare the quadratic form,

$$z^T L z = \sum_{(i,j)\in E} (z_i - z_j)^2,$$

with the sum of quadratic forms,

$$(z^+)^T L(z^+) + (z^-)^T L(z^-) = \sum_{(i,j)\in E} (z_i^+ - z_j^+)^2 + \sum_{(i,j)\in E} (z_i^- - z_j^-)^2,$$

edge by edge.

If both i, j belong to the support of z^+ or z^-, then the contribution is same in both equations. If one end, say i, belongs to z^+ and other to z^-, then contribution is $(z_i^+ - (-z_j^-))^2$ in the first equation but $(z_i^+)^2 + (z_j^-)^2$ for the second equation.

So, we get that,

$$z^T L z \geq (z^+)^T L z + (z^-)^T L z.$$

Now,

$$\begin{aligned} F(z) &= \frac{z^T L z}{z^T z} \\ &\geq \frac{(z^+)^T L z^+ + (z^-)^T L z^-}{z^T z} \\ &= \frac{(z^+)^T L z^+ + (z^-)^T L z^-}{\|z^+\|^2 + \|z^-\|^2} \\ &= \frac{\|z^+\|^2}{\|z^-\|^2 + \|z^-\|^2} F(z^+) + \frac{\|z^-\|^2}{\|z^-\|^2 + \|z^-\|^2} F(z^-) \\ &\geq \min(F(z^+), F(z^-)). \end{aligned}$$

Exercise 5. How do you prove the last inequality?

This prove Eqn. 3 and hence Lem. 2.

□

3.2 Analysis for special vectors

The analysis for special vectors is to look at a random threshold and show that the expected value of the expansion is small.

Lemma 3. *Let y be a non-negative vector. The expected value of expansion for the cuts obtained by rounding y can be bounded by,*

$$\frac{\mathbb{E}E(S_t, V - S_t)}{\mathbb{E}|S_t|} \leq \sqrt{2dF(y)},$$

where expectation is taken over t^2 which varies uniformly from 0 to the $T = \max_i y_i$.

Proof. Let the maximum coordinate of y be T, then t^2 varies from 0 to T. A vertex i is in S_t if $y_i^2 \geq t^2$,

$$Pr(i \in S_t) = \frac{y_i^2}{T}.$$

Then,

$$\mathbb{E}|S_t| = \sum_i Pr(i \in S_t) = \sum_i \frac{y_i^2}{T}.$$

Also, an edge is present in the cut $S_t, V - S_t$ only if t^2 lies between y_i^2 and y_j^2,

$$Pr((i,j) \in E(S_t, V - S_t) = \frac{\left|y_i^2 - y_j^2\right|}{T}.$$

So,

$$\mathbb{E}E(S_t, V - S_t) = \sum_{(i,j)\in E} Pr((i,j) \in E(S_t, V - S_t)) = \sum_{(i,j)\in E} \frac{\left|y_i^2 - y_j^2\right|}{T}.$$

So, we need to bound,

$$Q := \frac{\mathbb{E}|E(S_t, V - S_t)|}{\mathbb{E}|S_t|} = \frac{\sum_{(i,j)\in E} \left|y_i^2 - y_j^2\right|}{\sum_i y_i^2}.$$

The numerator of Q can be upper bounded by Cauchy-Schwarz,

$$Q \leq \frac{\sqrt{\sum_{(i,j)\in E} (y_i - y_j)^2}\sqrt{\sum_{(i,j)\in E} (y_i + y_j)^2}}{\sum_i y_i^2}.$$

Using the fact that $(y_i + y_j)^2 \leq 2(y_i^2 + y_j^2)$,

$$Q \leq \frac{\sqrt{\sum_{(i,j)\in E} (y_i - y_j)^2}\sqrt{2d \sum_i y_i^2}}{\sum_i y_i^2}.$$

So,

$$Q \leq \frac{\sqrt{2d(\sum_{(i,j)\in E} (y_i - y_j)^2)}}{\sqrt{\sum_i y_i^2}} \leq \sqrt{2dF(y)}.$$

proving the result.

□

Since $|S_t|$ is a strictly positive random variable. Using Thm. 2 (proof is given as an assignment), we can show that,

$$Pr\left(\frac{E(S_t, V - S_t)}{|S_t|} \leq \frac{\mathbb{E}E(S_t, V - S_t)}{\mathbb{E}|S_t|}\right) > 0.$$

Theorem 2. *Let $a_1, \cdots, a_n$ be non-negative and $b_1, \cdots, b_n$ be strictly positive. Then,*

$$\frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} \geq \min_i \frac{a_i}{b_i}.$$

Exercise 6. Prove the above inequality using Thm. 2.

Hence, there exists a t, s.t.,

$$\phi(S_t) \leq \sqrt{2dF(y)}.$$

3.3 Proof of Lem. 1

In last two sections, we have proved that any x orthogonal to $\mathbf{1}$ can be converted in to a y, such that,

- y has less than $\frac{|V|}{2}$ non-zero entries,
- objective value $F(y)$ is less than $F(x)$,
- cuts obtained by rounding on y are also cuts obtained by rounding on x,
- rounding on y will give a cut S with at most $|V|/2$ size, s.t.,

$$\phi(S) \leq \sqrt{\frac{2F(y)}{d}} \leq \sqrt{\frac{2F(x)}{d}}.$$

First three properties follow from Lem. 2 and the last property follows from Lem. 3. These four properties imply Lem. 1 and hence Thm. 1.

4 Assignment

Exercise 7. Define the normalized Laplacian to be,

$$N_G := D_G^{-1/2} L_G D_G^{-1/2} = I - D_G^{-1/2} A_G D_G^{-1/2}.$$

Let μ_1 be the smallest eigenvalue of N_G, show that,

$$\mu_1 = \min x \frac{x^T L_G x}{x^T D_G x}.$$

Exercise 8. Take the projection of $\mathbf{1}_S$ on $\mathbf{1}$, and subtract it out from $\mathbf{1}_S$. Show that you will get a scalar multiple of the following vector x.

$$x_i = \begin{cases} |S| & i \notin S \\ -|V - S| & i \in S \end{cases}$$

Exercise 9. Show that if a cut is obtained by choosing threshold t in rounding of z^+(or z^-), it is also obtained for some other threshold t' and rounding on x.

Exercise 10. Prove the Thm. 2.