# Lecture 12: Larget eigenvalue and coloring

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We will start exploring connections between eigenvalues of Laplacian and combinatorial properties of the graph from this lecture. In today's lecture, we look at the connection of eigenvalues with bipartiteness and coloring. We will finish by showing interlacing property of eigenvalues when an edge is deleted.

Like last lecture, assume that $L_G$ is the Laplacian of a graph $G = (V, E)$. Remember that $L_G$ is symmetric and hence has $n$ real eigenvalues. Let us call them, in the *ascending* order, $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.

Similarly, assume that $A_G$ is the adjacency matrix of a graph $G = (V, E)$. Remember that $A_G$ is symmetric and hence has $n$ real eigenvalues. Let us call them, in the *descending* order, $\mu_1 \geq \mu_2 \leq \cdots \geq \mu_n$.

We ordered the eigenvalues of the Laplacian and the adjacency matrix this way because these eigenvalues are related for a $d$-regular graph.

*Exercise 1.* Show that $\lambda_k = d - \mu_k$, for a $d$-regular graph.

If the graph is not connected, then the adjacency (or Laplacian) matrix is block diagonal. Hence, the eigenvalues of the complete graph is just the collection of eigenvalues of all the connected components. So, we assume that our graph is connected.

To remind you, a $k$-coloring of a graph is a map $C : v \to [k]$, such that, no two adjacent vertices have the same color,

$$(i, j) \in E \Rightarrow C(i) \neq C(j).$$

We are generally interested in the minimum number of colors needed to color a graph.

*Exercise 2.* Show that if you can color $G$ with $k$ colors then you can color it with $l > k$ colors too.

The minimum number of colors needed to color a graph $G$ is called the *chromatic number*, $\chi(G)$, of the graph.

*Exercise 3.* Show that the chromatic number of a bipartite graph is 2.

Finding the chromatic number of a graph is an NP-hard problem.

## 1 Perron Frobenius theorem

We know that every entry of an adjacency matrix $A_G$ is non-negative. Perron Frobenius theorem gives us,

- the maximum eigenvalue of $A_G$ has the largest absolute value, i.e., $\mu_1 \geq |\mu_n|$,
- the largest eigenvector is entry-wise non-negative.

*Proof.* You will prove the second property in the assignment. For the first property, we only need to show that for all unit vectors $v$,

$$\left|v^T A v\right| \leq \mu_1.$$

Let $u$ be the vector with absolute values of $v$ as coordinates, $u_i = |v_i|$. Then,

$$\left|v^T A v\right| = \left|\sum_{(i,j)\in E} A_{(i,j)} v_i v_j\right| \leq \left|\sum_{(i,j)\in E} A_{(i,j)} u_i u_j\right| = u^T A u.$$

This follows because $A_{(i,j)}$ is non-negative. But we know that the maximum value of $x^T A x$ is $\mu_1$ for all unit vectors $x$. So,

$$u^T A u \leq \mu_1.$$

□

If the graph is $d$-regular, we can convert this statement for the eigenvalues of the Laplacian.

*Exercise 4.* Show that the maximum eigenvalue of $L_G$ is less than or equal to $2d$.

## 2 Bipartiteness and $\lambda_n$

Let us assume that we have a $d$-regular graph $G = (V, E)$. From the previous discussion we know that $\lambda_n \leq 2d$.

It turns out that $\lambda_n$ captures the bipartiteness of the graph. In other words, we can prove the following theorem.

**Theorem 1.** *Given a connected $d$-regular graph $G = (V, E)$, the largest eigenvalue $\lambda_n$ is $2d$ if and only if $G$ is bipartite.*

*Proof.* We will prove the easy direction first. By definition of bipartite, we can divide the vertex set $V$ into two parts, say $A$ and $B$, such that, there are no edges insides $A$ and inside $B$. Arrange the vertices so that all vertices of $A$ come before all vertices of $B$. Then, $A_G$ looks like,

$$\begin{pmatrix} 0 & A_1 \\ A_2 & 0 \end{pmatrix}$$

Here, every row of $A$ and $B$ sum to $d$. Then, the vector $v$, such that,

$$v_i = \begin{cases} 1 & i \in A \\ -1 & i \in B, \end{cases}$$

is an eigenvector of $A_G$ with eigenvalue $-d$. In other words, there is an eigenvector of $L_G$ with eigenvalue $2d$.

*Exercise 5.* Prove the above statement.

For the other direction, we need to prove that $\lambda_n < 2d$ if the graph is not bipartite. We will prove that $M = 2dI - L_G$ is positive semidefinite and there is an eigenvalue 0 then the graph is bipartite.

We will use the quadratic form of $L$ for a unit vector $x$,

$$x^T M x = x^T (2dI)x - x^T L_G x = 2d \sum_i x_i^2 - x^T L_G x. \tag{1}$$

From the previous lecture, we know the quadratic form of $L_G$,

$$x^T L_G x = \sum_{(i,j) \in E} (x_i - x_j)^2 = \sum_i d x_i^2 - 2 \sum_{(i,j) \in E} x_i x_j.$$

Substituting this value in Eqn. 1,

$$x^T M x = d \sum_i x_i^2 + \sum_{(i,j) \in E} x_i x_j = \sum_{(i,j) \in E} (x_i + x_j)^2.$$

This shows that $M$ is positive semidefinite and the maximum eigenvalue of $L_G$ could be at most $2d$.

The quadratic form, $x^T M x$, can only be zero if $x_i = -x_j$ whenever $(i, j) \in E$. So, the negative and positive coordinates of $x$ give the bipartition of $G$.

□

## 3 Coloring and eigenvalues of the adjacency matrix

Most of the content of this section is taken from Dan Spielman's course notes.

Let us first see the connection between $\mu_1$ and degrees in the graph. For a $d$-regular graph, we know that $\mu_1 = d$. You will show in the assignment that $\mu_1$ is less than the maximum degree for a general graph.

**Theorem 2.** *Define $d'$ to be the average degree of a graph $G$. Then,*

$$\mu_1 \geq d'.$$

*Let $d''$ to be the maximum average degree over any subgraph of $G$. Even then,*

$$\mu_1 \geq d''.$$

*Proof.* Again, we will use the fact that $\mu_1$ maximizes the quadratic form $x^T A_G x$ over all unit vectors $x$. In other words, $x^T Ax$ for any unit vector $x$ is a lower bound on $\mu_1$.

To prove the first assertion, choose $x$ to be,

$$\frac{1}{\sqrt{n}}(1\ 1 \cdots 1).$$

The quadratic form becomes $\sum_{i,j} A_{i,j} x_i x_j = \frac{1}{n} \sum_{i,j} A_{i,j}$.

*Exercise 6.* Show that $x^T Ax$ is the average degree of graph $G$.

This proves that $d'$ is a lower bound on $\mu_1$.

To prove that $d''$ is a lower bound on $\mu_1$, we will show that for every subgraph of $G$, average degree is a lower bound on $\mu_1$. This lower bound is achieved by using $x$ to be the normalized indicator vector for the subgraph of $G$.

*Exercise 7.* Convince yourself that if $x_S$ is the indicator vector for a subgraph $S$ of $G$, then $x_S^T A_G x_s$ is the average degree of $S$.

□

Coming back to coloring, you will show in the assignment that you can always color a graph using *maximum degree plus one* number of colors. If there is a clique of size $k$ in the graph then you need at least $k$ colors.

*Exercise 8.* Show that if a graph $G$ has a clique of size $k$ then $\mu_1 \geq k - 1$.

The next theorem, known as Wilf's theorem, shows that you can color a graph with $\lfloor \mu_1 \rfloor + 1$ colors.

**Theorem 3 (Wilf).** *Let $G$ be a graph with adjacency matrix $A_G$ and $\mu_1$ be the maximum eigenvalue of $A_G$. Then, $G$ can be colored with $\lfloor \mu_1 \rfloor + 1$ colors.*

*Proof.* We will prove the theorem by induction. Since $\mu_1$ is more than the average degree by Thm. 2, there exists a vertex with less than or equal to $\lfloor \mu_1 \rfloor$ number of neighbours. Call this vertex $v$.

Look at the subgraph $G'$ on vertex set $V - v$. We will show that it can be colored with $\lfloor \mu_1 \rfloor + 1$ number of colors. Then, vertex $v$ can also be colored with these colors. This is because $v$'s neighbours are one less than the total number of colors, leaving one color for the vertex $v$ itself.

To prove that $G'$ can be colored with $\lfloor \mu_1 \rfloor + 1$ colors, using induction, we only need to prove that the maximum eigenvalue of $A_{G'}$ is less than the maximum eigenvalue of $A_G$.

This follows simply because $A_{G'}$ is a submatrix of $A_G$. This fact is given as an assignment. □

## 4 Eigenvalues with edge deletion

Let $L_G$ be the Laplacian of a graph $G$. We are interested in the eigenvalues of $L_{G'}$ where $G'$ is the graph with one edge deleted from $G$.

*Note 1.* Previously we talked about subgraph when a vertex was deleted. This time, we are deleting an edge.

We will use Courant-Fischer theorem, covered in the last class, to relate the eigenvalues of $L_G$ and $L_{G'}$. For your reference, the theorem is given below.

**Theorem 4 (Courant-Fischer).** *Let $M$ be a symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$. Then,*

$$\lambda_k = \min_{S:dim(S)=k} \max_{x \in S:\|x\|=1} x^T M x.$$

*Here, $S$ is a subspace of $\mathbb{R}^n$. Also,*

$$\lambda_k = \max_{S:dim(S)=n-k+1} \min_{x \in S:\|x\|=1} x^T M x.$$

Suppose we delete an edge $(i,j)$ from $G$ to obtain $G'$. Say, $v$ be the vector with 1 at $i$-th position, $-1$ at the $j$-th position and 0 otherwise. The important thing to notice is that $L_G = L_{G'} + vv^T$.

*Exercise 9.* Prove the above statement.

The following theorem gives the relationship between the eigenvalues of $M$ and $M' = M \pm vv^T$, addition/subtraction by a rank one matrix, using the Courant-Fischer theorem.

**Theorem 5.** *Let $\lambda_1 \le \lambda_2 \cdots \le \lambda_n$ be the eigenvalues of $M$ and $\eta_1 \le \eta_2 \cdots \le \eta_n$ be the eigenvalues of $M' = M \pm vv^T$ for some vector $v$. Then,*

$$\lambda_i \le \eta_{i+1}.$$

Before we start the proof, we need a lemma about subspaces.

**Lemma 1.** *Suppose we are given a subspace $S$ of dimension $k$ and a vector $v$. For any vector $x \in S$ and $x \perp v$, we can always find a subspace $S'$ of dimension $k-1$, such that,*

$$x \in S' \text{ and } v \perp S'.$$

*Proof.* If $v \perp S$ then any $S'$ containing $x$ will work.

Otherwise, let $v_1, \cdots v_n$ be an orthonormal basis of the entire space, such that, $v_1, \cdots, v_k$ is a basis of $S$ and $v_{k+1}, \cdots, v_n$ are perpendicular to $S$.

Then, $v$ can be written as $v = \sum_{i=1}^n \alpha_i v_i$. Now, construct a basis $w_1, \cdots, w_k$ by Gram-Schmidt decomposition, such that, $w_1$ is the unit vector in the direction of $\sum_{i=1}^k \alpha_i v_i$.

The following exercise completes the proof.

*Exercise 10.* Show that the space spanned by $w_2, \cdots, w_k$ is the required subspace $S'$. That is, $x \in span(w_2, \cdots, w_k)$ and $v \perp w_i$ for all $i \ne 1$.

□

*Proof of Thm. 5.* From Courant-Fishcher theorem,

$$\eta_{i+1} = \min_{S:dim(S)=i+1} \max_{x \in S:\|x\|=1} x^T (M + vv^T) x.$$

If we restrict the feasible space of a maximization problem, the optimal value decreases.

$$\eta_{i+1} \ge \min_{S:dim(S)=i+1} \max_{x \in S:\|x\|=1, x \perp v} x^T (M + vv^T) x = \min_{S:dim(S)=i+1} \max_{x \in S:\|x\|=1, x \perp v} x^T M x$$

Look at the space $S$ which minimizes the right hand side of the previous equation with vector $x$. Let $S' \subseteq S$ be the subspace of dimension $i$ form Lem. 1, such that, $v$ is perpendicular to $S'$ and $x$ is in $S'$. This shows that we can just optimize over all subspaces $S'$ of dimension $i$ with $v$ being perpendicular to $S'$.

$$\eta_{i+1} \geq \min_{S':dim(S')=i, v \perp S'} \max_{x \in S':\|x\|=1} x^T M x.$$

Again, we notice that if feasible space is increased than we get a higher value for a minimization problem.

$$\eta_{i+1} \geq \min_{S':dim(S')=i} \max_{x \in S':\|x\|=1} x^T M x = \lambda_i.$$

The last equality follows from Courant-Fischer theorem. □

By interchanging $M$ and $M'$, Thm. 5 shows that $\eta_i \leq \lambda_{i+1}$. Hence, Thm. 5 implies that the eigenvalues of $L_G$ and $L_{G'}$ are interlaced with each other.

## 5 Assignment

*Exercise 11.* Suppose $M'$ is a submatrix of $M$. Show that the maximum eigenvalue of $M'$ is less than the maximum eigenvalue of $M$. Similarly, show that the minimum eigenvalue of $M'$ is bigger than the minimum eigenvalue of $M$.

*Exercise 12.* Suppose $A_G$ is the adjacency matrix of a connected graph. Show that the eigenvector corresponding to the highest eigenvalue is non-negative. Then, show that it is strictly positive.

*Exercise 13.* Show that if a graph $G$ is not bipartite then $\mu_1 = -\mu_n$.

*Exercise 14.* Show that the largest eigenvalue of the adjacency matrix is smaller than the maximum degree.

*Exercise 15.* Let $d_m$ be the maximum degree in a graph. Prove that you can color a graph with $d_m + 1$ colors.