Lecture 5: Quotient group

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We have seen that the cosets of a subgroup partition the entire group into disjoint parts. Every part has the same size and hence Lagrange's theorem follows. If you are not comfortable with cosets or Lagrange's theorem, please refer to earlier notes and refresh these concepts.

So we have information about the size of the cosets and the number of them. But we lack the understanding of their structure and relations between them. In this lecture, the concept of *normal subgroups* will be introduced and we will form a group of cosets themselves !!

1 Normal subgroup

Suppose we are given two elements g, n from a group G. The *conjugate* of n by g is the group element gng^{-1}.

Exercise 1. When is the conjugate of n equal to itself?

Clearly the conjugate of n by g is n itself iff n and g commute.

We can similarly define the conjugate of a set $N \subseteq G$ by g,

$$gNg^{-1} := \{gng^{-1} : \ n \in N\}.$$

Definition 1. *Normal subgroup: A subgroup N of G is* normal *if for every element g in G, the conjugate of N is N itself.*

$$gNg^{-1} = N \quad \forall g \in G.$$

We noticed that $gng^{-1} = n$ iff g, n commute with each other.

Exercise 2. When is $gNg^{-1} = N$?

In this case the left and right cosets are the same for any element g with respect to subgroup N. Hence, a subgroup is normal if its left and right cosets coincide.

Exercise 3. Show that following are equivalent. So you need to show that each of them applies any other.

1. N is a normal subgroup.
2. The set $S = \{g : gN = Ng\}$ is G itself.
3. For all elements $g \in G$, $gNg^{-1} \subseteq N$.

Hint: Instead of showing all $2 \times \binom{3}{2}$ implications, you can show $1) \Rightarrow 2) \Rightarrow 3) \Rightarrow 1$.

2 Quotient group

We have introduced the concept of normal subgroups without really emphasizing why it is defined. Lets move to our original question. What can be said about the set of cosets, do they form a group?

Suppose G is a group and H is a subgroup. Denote by S, the set of cosets of G with respect to H. For S to be a group it needs a law of composition. The most natural composition rule which comes to mind is,

$$(gH)(kH) = (gk)H.$$

Here gH and kH represent two different cosets. The problem with this definition is that it might not be *well-defined*. It might happen that $g' \in gH$ and $k' \in kH$ when multiplied give a totally different coset $(g'k')H$ then $(gk)H$.

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[*] Thanks to the book from Dummit and Foote and the book from Norman Biggs.

Exercise 4. Show that this operation is well-defined for commutative (abelian) groups.

What about the general groups? Here comes the normal subgroup to the rescue.

Theorem 1. *Suppose G is a group and H is its subgroup, the operation,*

$$(gH)(kH) = (gk)H,$$

is well defined if and only if H is a normal subgroup.

Note 1. Every subgroup of a commutative group is normal.

Proof. $\Rightarrow$): We need to show that if the operation is well defined then $ghg^{-1} \in H$ for every $g \in G, h \in H$. Consider the multiplication of H with $g^{-1}H$. Since $e, h \in H$, we know $eH = hH$. Since the multiplication is well defined,

$$(eg^{-1})H = (eH)(g^{-1}H) = (hH)(g^{-1}H) = (hg^{-1})H \quad \Rightarrow \quad g^{-1}H = (hg^{-1})H.$$

Again using the fact that $e \in H$, $hg^{-1} \in g^{-1}H$. This implies $hg^{-1} = g^{-1}h' \quad \Rightarrow ghg^{-1} = h'$ for some $h' \in H$.

$\Leftarrow$): Suppose N is a normal subgroup. Given $g' = gn$ and $k' = kn'$, where $g, g', k, k' \in G$ and $n, n' \in N$, we need to show that $(gk)N = (g'k')N$.

$$(g'k')N = (gnkn')N.$$

Exercise 5. Show that there exist $m \in N$, s.t., $nk = km$. Hence complete the proof.

$\square$

With this composition rule we can easily prove that the set of cosets form a group (exercise).

Definition 2. *Given a group G and a normal subgroup N, the group of cosets formed is known as the* quotient group *and is denoted by $\frac{G}{N}$.*

Using Lagrange's theorem,

Theorem 2. *Given a group G and a normal subgroup N,*

$$|G| = |N||\frac{G}{N}|$$

3 Relationship between quotient group and homomorphisms

Let us revisit the concept of homomorphisms between groups. The homomorphism between two groups G and H is a mapping $\phi : G \to H$ that preserves composition.

$$\phi(gg') = \phi(g)\phi(g')$$

For every homomorphism ϕ we can define two important sets.

- Image: The set of all elements h of H, s.t., there exists $g \in G$ for which $\phi(g) = h$.

$$Img(\phi) = \{h \in H : \ \exists g \in G \ \ \phi(g) = h\}$$

Generally, you can restrict your attention to $Img(\phi)$ instead of the entire H.

– Kernel: The set of all elements of G which are mapped to identity in H.

$$Ker(\phi) = \{g \in G : \quad \phi(G) = e_H\}$$

Notice how we have used the subscript to differentiate between the identity of G and H.

Note 2. $Img(\phi)$ is a subset of H and $Ker(\phi)$ is a subset of G.

Exercise 6. Prove that $Img(\phi)$ and $Ker(phi)$ form a group under composition with respect to H and G respectively.

Exercise 7. Show that $Ker(\phi)$ is a normal subgroup.

There is a beautiful relation between the quotient groups and homomorphisms. We know that $Ker(\phi)$ is the set of elements of G which map to identity. What do the cosets of $Ker(\phi)$ represent. Lets take two elements g, h of a coset $gKer(\phi)$. Hence $h = gk$ where $\phi(k) = e_H$. Then by the composition rule of homomorphism $\phi(g) = \phi(h)$.

Exercise 8. Prove that $\phi(g) = \phi(h)$ if and only if g and h belong to the same coset with respect to $Ker(\phi)$.

The set of elements of G which map to the same element in H are called the fibers of ϕ. The previous exercise tell us that fibers are essentially the cosets with respect to $Ker(\phi)$ (the quotient group).

The fibers are mapped to some element in $Img(\phi)$ by ϕ. Hence there is a one to one relationship between the quotient group $\frac{G}{Ker(\phi)}$ and $Img(\phi)$. Actually the relation is much stronger.

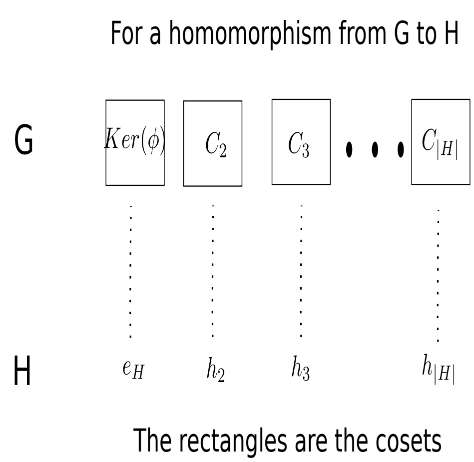

Fig. 1. Relationship between the quotient group and the image of homomorphism

It is an easy exercise to show that the mapping between quotient group $\frac{G}{Ker(\phi)}$ and $Img(\phi)$ is an isomorphism.

Exercise 9. Prove that the above mapping is an isomorphism.

Again applying the Lagrange's theorem,

$$|G| = |Ker(\phi)||Img(\phi)|.$$

The figure 2 depict that every element of quotient group is mapped to one element of image of ϕ. Now we know that this mapping is *well-behaved* with respect to composition too.

$$\phi((gKer(\phi))(hKer(\phi))) = \phi(gKer(\phi))\phi(hKer(\phi))$$

There is an abuse of notation which highlights the main point also. The notation $\phi(gKer(\phi))$ represents the value of ϕ on any element of $gKer(\phi)$. We know that they all give the same value. The study of homomorphism is basically the study of quotient group. The study of quotient group can be done by choosing a representative for every coset and doing the computation over it (instead of the cosets).

We have shown that $Ker(\phi)$ is normal. It can also be shown that any normal subgroup N is a kernel of some homomorphism ϕ (exercise).

4 Assignment

Exercise 10. Given a subgroup H of G, two elements $x, y \in G$ are related ($x \sim y$) if $x^{-1}y \in H$. Prove that this relation is an equivalence relation. What are the equivalence classes of this relation?

Exercise 11. Given a group G and a normal subgroup N. Say the set of cosets is called S and has composition operation $(gH)(kH) = (gk)H$. Show that,

- Identity exists in this set.
- Inverses exist in this set.
- Associativity is satisfied.

Since Closure is obvious we get that S is a group with respect to the above mentioned composition rule.

Exercise 12. Given a group G and a subgroup N as a set. Write a program to find if N is normal or not. Assume that you are given a function $mult(x, y)$, which can compute the binary operation of the group G between any two elements x, y of G.

Exercise 13. What is the quotient group of D_{2n} with respect to the subgroup generated by reflection?

Exercise 14. Suppose G is an abelian group and H is a subgroup. Show that $\frac{G}{H}$ is abelian.

Exercise 15. Given N is a normal subgroup, prove that $g^k(N) = (gN)^k$.

Exercise 16. Suppose N is normal in G, show that for a subgroup H, $H \cap N$ is a normal subgroup in H.

Exercise 17. Show that a subgroup N is normal in G iff it is the kernel of a homomorphism from G to some group H.