Lecture 4: Orbits

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In the beginning of the course we asked a question. How many different necklaces can we form using 2 black beads and 10 white beads? In the question, the numbers 2 and 10 are arbitrarily chosen. To answer this question in a meaningful way, we need to construct a strategy or theorem which will answer the above question for any such numbers. But to understand the question better, lets ask a simpler question.

Exercise 1. How many different necklaces can be formed using 2 white and 2 black beads?

Lets look at the question in detail. The first guess for the question would be $4! = 24$, the number of ways we can permute the four beads. But all these permutations need not be different. What do we mean by *different* necklaces? It might happen that two different permutations (σ_1 and σ_2) might be the same in the sense that σ_1 can be obtained from σ_2 using rotation. Look at figure 1 for one such example.

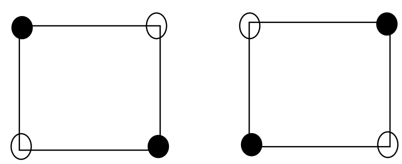

Fig. 1. Two permutations giving the same necklace

Using some brute force now, we can come up with all possible different necklaces for 2 white and 2 black beads (figure 1).

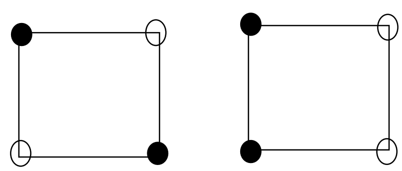

Fig. 2. Possible different necklaces with 2 white and 2 black beads

You can convince yourself that the question is much harder if we take bigger numbers. What should we do? When are two necklaces equivalent?

Two necklaces are equivalent if we can obtain one by applying a symmetry to other necklace (like rotation or reflection). We know from discussions in previous classes that the symmetries form the dihedral group, D_{2n}. The strategy would be to develop a general framework for groups to answer question about distinct necklaces.

1 Group action

The first thing to notice in the necklace problem is that there are two different objects of interest. One is the set of necklaces (set of all permutations of the necklaces) and other is the set of symmetries. A symmetry can be applied to a necklace to obtain another necklace. Lets make this action abstract.

* Thanks to the book from Dummit and Foote and the book from Norman Biggs.

Abstractly, given a group G and a set A, every element g of G *acts* on set A. That means for every element g there is a function from A to A which is called its action on G. For the sake of brevity, we will denote the function corresponding to the element $g \in G$ with g itself. Hence the value of $a \in A$ after action of g will be called $g(a)$.

Exercise 2. What is the group and what is the set for the necklace problem?

Note 1. It is NOT the set of distinct necklaces.

Lets look at the formal definition.

Definition 1. *Given a group G and a set A, a group action from G to A assigns a function $g : A \to A$ for every element g of group G. A valid group action satisfies the following properties.*

- *Identity takes any element $a \in A$ to a itself, i.e., $e(a) = a$ for every $a \in A$.*
- *For any two group elements $g_1, g_2 \in G$, their functions are consistent with the group composition,*

$$g_1(g_2(a)) = (g_1 g_2)(a).$$

Using this definition and group structure of G, it can be shown that action of g is a permutation on the elements of A.

This gives us another representation of group elements. For any group action on A of size m, we have a permutation representation for any element $g \in G$ in terms of a permutation on m elements. In the following sections we will keep this representation in mind.

Note 2. Actually a slightly stronger theorem holds. It is called *Cayley's theorem* and is given below. We will not show the proof of this theorem.

Theorem 1. *Cayley's theorem: Every group of order n is isomorphic to some subgroup of S_n.*

2 Orbits

Suppose we are given action of group G on a set A. Lets define a relation between the elements of A. If $\exists\ g :\ g(x) = y$ then we will say that x, y are related ($x \sim y$). We can easily prove that this relation is equivalence relation.

- Reflexive: Why?
- Symmetric: Suppose $x \sim y$ because $g(x) = y$. Then consider $x = ex = (g^{-1}g)(x) = g^{-1}y$, implying $y \sim x$.
- Transitive: Show it as an exercise.

Hence this equivalence relation will partition the set A into distinct equivalence classes. The equivalence class corresponding to $x \in A$ is the orbit ($G(x)$) of element x. In other words,

$$G(x) = \{g(x) :\ g \in G\}.$$

Now we will look at two counting questions,

1. What is the size of these orbits?
2. How many distinct orbits are there?

Why are we interested in these questions. Let us look at this concept from the example of necklaces. If a necklace x can be obtained from another necklace y using a symmetry then they are related (in the necklace case indistinguishable).

Exercise 3. Convince yourself that the number of distinct necklaces is the same as the number of distinct orbits (equivalence classes) under the dihedral group D_{2n}.

We will answer both the counting questions under the general group-theoretic framework. As a special case, this will solve the necklace problem.

2.1 stabilizers

Remember that the orbit of $x \in A$ under the action of G can be defined as,

$$G(x) = \{g(x): \ g \in G\}.$$

If every $g \in G$ took x to a different element, the size of the orbit would be $|G|$. But this is too much to expect. If we consider any example, there will be lots of $g \in G$ which will take x to a single element y. Lets define this set as G(x,y),

$$G(x, y) = \{g \in G: \ g(x) = y\}.$$

Exercise 4. Does the set $G(x, y)$ form a subgroup of G? Under what condition will it form a subgroup?

The answer to the previous exercise is when $x = y$. The set $G_x := G(x, x)$ is called the stabilizer of x,

$$G_x = \{g \in G: \ g(x) = x\}.$$

Exercise 5. If you were not able to solve the previous exercise, prove that G_x is a subgroup of G.

Once we have the subgroup G_x, the natural question to ask is, what are the cosets? This is where we get lucky. Suppose y is an element of the orbit $G(x)$. So there exist an $h \in G$, s.t., $h(x) = y$. Then $G(x, y)$ is precisely the coset hG_x.

Lemma 1. *Given a $y \in A$, s.t., $h(x) = y$. The coset hG_x is same as the set $G(x, y)$.*

Proof. $\Rightarrow$: An element of hG_x is of the form $hg, \ g \in G_x$. Then $hg(x) = h(x) = y$. So $hG_x \subseteq G(x, y)$.
$\Leftarrow$: Suppose $g \in G(x, y)$, i.e., $g(x) = y$. Then show that,

Exercise 6. $h^{-1}g \in G_x$

But $h^{-1}g \in G_x$ implies $g \in hG_x$.

Exercise 7. Show the above implication. Be careful, It is not just the same as multiplying by h on both sides.

From the previous exercise $G(x, y) \subseteq G_x$. $\square$

Hence, for every element y in the orbit $G(x)$, there is a coset. It is an easy exercise to convince yourself that every coset will correspond to a single element in the orbit $G(x)$. So the number of elements in the orbit is equal to the number of cosets. But we know that the number of cosets can be calculated from Lagrange's theorem. Hence,

$$|G| = |G_x||G(x)|.$$

Note 3. G_x is a subset of G, but $G(x) \subseteq A$. The equation works because we show a one to one relation between $G(x)$ (orbit) and the cosets.

2.2 Burnside's lemma

We now know the size of the orbit. Given an element x with stabilizer G_x, the number of elements in its orbit is $\frac{|G|}{|G_x|}$. Can this help us in counting the number of distinct orbits.

Lets give every element on A a weight of $\frac{1}{|G(x)|}$. The number of distinct orbits is now the sum of weights of all elements of A.

$$\text{Number of distinct orbits } = \sum_{x \in A} \frac{1}{|G(x)|} = \frac{1}{|G|} \sum_{x \in A} |G_x|.$$

Lets concentrate on the summation. The total summation is equal to the number of pairs $(g \in G, x \in A)$, s.t., $g(x) = x$. Suppose we make a matrix with rows indexed by elements of G and columns indexed by elements of A. The entry (g, x) is one if $g(x) = x$ and 0 otherwise.

$$\begin{array}{c|cccc} & x_1 & x_2 & \cdots & x_{|A|} \\ \hline g_1 & 0 & 1 & \cdots & 1 \\ g_2 & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_{|G|} & 1 & 0 & \cdots & 0 \end{array}$$

Then $\sum_x |G(x)|$ is the number of 1's in the matrix above. Each term in the summation, $|G_x|$ is the number of 1's in the column corresponding to x. We can count the number of 1's in the matrix by taking the sum row-wise too. Suppose, $S(g)$ is set of elements of A fixed by g.

$$S(g) = \{x : \; g(x) = x, \; x \in A\}.$$

Using $S(g)$ we get the orbit-counting (Burnside's) lemma.

Lemma 2. *Burnside's lemma (Orbit-counting): Given a group action of G over A. The number of distinct orbits can be written as,*

$$\textit{Number of distinct orbits } = \frac{1}{|G|} \sum_{g \in G} |S(g)|.$$

Note 4. The summation is now over G instead of A.

One natural question you might ask is, How did this benefit us? Previously we were summing over all possible $x \in A$ and now we are summing up over all $g \in G$. The reason is, in general, the size G will be much smaller than size of A.

Lets look at an example where Orbit-counting lemma will help us in answering the question about necklaces. Try to solve this exercise yourself first and later you can look at the solution given below.

Exercise 8. How many necklaces can be formed with 2 black and 6 white beads?

Arrange the beads on the vertices of a regular 8-gon. Since the necklaces are obtained by fixing the position of 2 black beads, there are 28 elements in A. The symmetry group is D_{16} with 16 elements.

For different elements of G we can calculate the number of elements fixed by it.

- Identity e: Fixes 28 elements.
- Out of 7 other rotations, only one of them fixes 4 elements. What is the angle of that rotation? Rest do not fix anything.
- All reflections fix exactly 4 elements. It is easy to see by looking at the cycle structure of the permutation. All beads of the same color should fall in the same cycle.

So by Orbit-counting lemma,

$$\text{Number of distinct orbits } = \frac{1}{16}(28 + 4 + 8 \times 4) = 4.$$

Exercise 9. What are the four configurations? Can you characterize them?

For other examples of application of Burnside's lemma, please look at section 21.4 of Norman Biggs book. There is a nice example in Peter Cameron's notes on Group Theory too (section 1.3).

3 Group representations (advanced)

We looked at the permutation representation of every group. There is a matrix representation of every group too. The study of that representation is called the group representation theory. Group representation theory is one of the main tools to understand the structure of a group. We will only give a very basic idea of this field. Interested students can look at book *Algebra* by Artin.

Define GL_n to be the group of invertible matrices of size $n \times n$ with complex entries. We can also think of these matrices as linear operators over $\mathbb{C}^n$.

A linear representation (matrix representation) is a homomorphism from group G to the group GL_n (say $R : G \to GL_n$). That means, we map every element of group G to an invertible matrix, s.t., it obeys the group composition,

$$R(gh) = R(g)R(h).$$

If there is a subspace of $\mathbb{C}^n$ which is fixed by every group element, then the representation is reducible. In other words, for an irreducible representation, there is NO subspace which is fixed by every element of group G.

Exercise 10. What does it mean that the subspace is fixed?

It can be shown that every representation can be broken down into irreducible representations. Another quantity of interest is the *character*. The character of the representation is a function $\chi : G \to \mathbb{C}$ defined by $\chi(g) = trace(R(g))$.

The characters of irreducible representations are orthonormal to each other and satisfy various other nice properties. Many theorems in group theory are derived by studying the characters of the group. Again, interested students can find more information in the book *Algebra* by Artin.

4 Assignment

Exercise 11. Show that action of dihedral group on the set of necklaces is a group action.

Exercise 12. Show that G is isomorphic to a subgroup of $S_{|A|}$. Remember that S_n is the group of permutations of $[n]$.

Hint: First show that action of g on A is a permutation and then use the consistency of group action with the group composition.

Exercise 13. Suppose we want to find the number of necklaces with m black and n white beads. What is the size of G and what is the size of A in terms of m, n?

Exercise 14. Prove that the average number of elements fixed by an element of group G under group action is an integer.

Exercise 15. Prove that GL_n is a group. What composition rule did you use?

Exercise 16. Biggs: Let G be a group of permutations of set A. If u, v are two elements in the same orbit of G, show that $|G_u| = |G_v|$.

Exercise 17. Biggs: Let A denote the set of corners of the cube and let G denote the group of permutations of A which correspond to rotation of the cube. Show that,

- G has just one orbit.
- For any corner x, $|G_x| = 3$.
- $|G| = 24$

Exercise 18. Biggs: Suppose you manufacture an identity card by punching two holes in an 3×3 grid. How many distinct cards can you produce. Look at the figure given below.

Hint: The group to consider here is D_8.

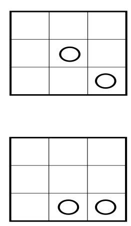

Fig. 3. Different identity cards with circles showing the holes.