Notes: Background on Analysis

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This is just a small note on basic concepts in functional analysis which will be used in the course. These definitions are taken from W. Rudin's book on functional analysis. For a detailed introduction to these concepts, please refer to Rudin's book or any elementary book on functional analysis. Most of the time we will talk about vector space $\mathbb{R}^n$, but these concepts are valid for general vector spaces.

1 Normed spaces

The vector space $\mathbb{R}^n$ is a normed space. Where the norm is,

$$\|x\| = \sqrt{x^T x}$$

This satisfies all the general definitions of being a norm.

- Triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$
- $\|\alpha x\| = |\alpha|\|x\|$, here $\alpha \in \mathbb{R}$
- $\|x\| \geq 0$.

This norm helps us in defining a metric space (intuitively, space where distances can be measured). The metric (distance) between two points x, y is $d(x, y) = \|x - y\|$. In this metric space, we can define the open ball of radius r centered at x,

$$B_r(x) = \{y : d(x, y) < r\}.$$

The closed ball of radius r is defined similarly,

$$B_r(x) = \{y : d(x, y) \leq r\}.$$

2 Open and closed sets

With these definitions, we can now define an open set and a closed set. Given a subset of points in $\mathbb{R}^n$ (say S), it is open, if for every point x in S there exist an open ball of non-zero radius $B_r(x)$ inside the set S.

For a general set S, a point x is on the boundary, if it is in S, but the set does not contain an open ball of any non-zero radius centered at x.

A set S is closed if its complement is an open set. Notice that there might exist sets which are neither closed nor open. In $\mathbb{R}^n$, this definition is equivalent to saying that the set is closed under the limit operation. So if $x_1, \cdots, x_k, \cdots \in S$ are a sequence of points, then their limit $x = \lim_{x \to \infty} x_i$ is also in set S.

Exercise 1. Give a set which is neither closed nor open.

The closure of a set S is the intersection of all closed sets which contain S. The interior of a set S is the union of all open sets inside S. For a general set, it will have some points on boundary and some points inside the set. The sets which don't have a boundary are the open sets.

3 Compact sets

A set S is bounded if it is contained in an open ball of finite radius. A set is compact if it is closed and bounded. This is not the original definition of compact sets and is not the right definition for general topological spaces. But for our purpose, in $\mathbb{R}^n$, this definition is true and much more useful.

The neighbourhood of any point x is any open set which contains x.