Lecture 7: Dual cones

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Convex optimization problems are interesting not just because they are easy to solve and lot of problems can be modeled in this domain. One of the very interesting aspect is the duality theory behind these optimization classes. Duality theory arises because of the concept of dual cones. So today we will focus on the concept of dual cones.

1 Supporting hyperplane theorem

We learned the separating hyperplane theorem last time. One corollary of this theorem is that for any point p on the boundary of a non-empty convex set C, there exists a supporting hyperplane at p on the set C. Here a supporting hyperplane at p is any a, s.t.,

$$a^T p \geq a^T x, \ \forall x \in C.$$

Geometrically, given a convex set and a point on the boundary of the convex set. It is always possible to draw a tangential hyperplane through the point so that the convex set lies only on one side of the hyperplane.

2 Dual cone

Given a cone C, we can define the *dual cone* by

$$C^* = \{y : y^T x \geq 0, \ \forall x \in C\}$$

Here $y^T x$ is the inner product on $\mathbb{R}^n$. In general, the same definition can be used in other spaces using their respective inner product.

Suppose $y \in C^*$. Then entire cone lies on one side of the hyperplane which is normal to y. The cone C contains origin (0). In other words, the hyperplane with normal y, passing through origin; is a supporting hyperplane at 0.

Given two hyperplanes which are supporting at 0, their convex combination (convex combination of normals) will also give a supporting hyperplane. Geometrically, the convex combination will lie *in between* the original two hyperplanes. Hence,

$$y_1, y_2 \in C^* \rightarrow \theta y_1 + (1-\theta) y_2 \in C^*, \ \forall \theta \in \mathbb{R}$$

It is again easy to see that

$$y \in C^*, \ \theta \geq 0 \Rightarrow \theta y \in C^*.$$

So the dual cone is a convex cone irrespective of whether the original cone was convex or not. Notice from the definition the dual cone is always closed too (all limits exist in the dual cone). Actually, C need not be a cone. For any set C, we can define the dual and it will be a convex cone.

Exercise 1. Can you describe, what will be the dual cone of a set C?

Remark: Polar cone is just the negative of a dual cone.

$$C' = \{y : y^T x \leq 0, \ \forall x \in C\}$$

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* Thanks to books from Boyd and Vandenberghe, Dantzig and Thapa, Papadimitriou and Steiglitz

2.1 Examples

Lets take a look at few examples and find out the dual of some cones.

- Subspace: First, show that a linear subspace L (all linear combinations are present in the set) is a cone. We leave it as an easy exercise.
  A vector y is in L^* iff $y^T x \geq 0, \ \forall x \in L$. But in a subspace, if x is present then so is $-x$. So $y^T x = 0, \ \forall x \in L$. Hence, the dual cone to the subspace is all vectors normal to it.
- Positive orthant ($\mathbb{R}^n_+$): We discussed that this set is a cone before and the usual $\geq$ inequality is the generalized inequality with respect to this cone.

  Exercise 2. Show that the positive orthant is the dual cone of itself

  Such cones are called self dual cones. We will see that the *positive semidefinite cone* (define it later), which will be really important later, is also a self dual cone.

Exercise 3. Show that is $C \subseteq D$, then $D^* \subseteq C^*$.

3 Dual cone of a closed and convex cone

By the definition of dual cone, we know that the dual cone C^* is closed and convex. Specifically, the dual of a closed convex cone is also closed and convex. First we ask what is the dual of the dual of a closed convex cone.

3.1 Dual of the dual cone

The natural question is what is the dual cone of C^* for a closed convex cone C. Suppose $x \in C$ and $y \in C^*$, then we know $y^T x \geq 0$. Since this condition is symmetric, we get that every element of C will be in C^{**}, i.e., $C \subseteq C^{**}$. We will show that these two sets are indeed equal.

For the other direction, we need to show that any element not in C has negative inner product with at least one $y \in C^*$. Suppose $x \notin C$, then by Farkas lemma, there exists a hyperplane which separates the point x and the cone C. So there exists y, s.t., $y^T x < 0 \leq y^T z, \ \forall z \in C$.

Then $y \in C^*$ by definition. We know, $y^T x < 0$, so $x \notin C^{**}$. Hence, for a closed convex cone C,

$$C = C^{**}.$$

Exercise 4. Dual cone can be defined for arbitrary sets also. What will be the dual of the dual in this case?

3.2 Dual of a proper cone

As discussed last time, we will mostly be interested in proper cones. Remember that they are convex, closed cones which do not contain any line and have nonempty interior.

Since C has nonempty interior, there exists a ball of non-zero radius inside the cone. Suppose the dual cone contains a line $\theta y, \ \theta \in \mathbb{R}$. Then for every point $x \in C$, $y^T x = 0$ (why). It implies that $y^T x = 0$ for the entire ball contained in the cone too. Since the inner product of y with center of the ball is zero and in every direction we move the inner product is zero. This implies the inner product with entire space is zero. Hence $y = 0$. So C^* does not contain a line.

Similarly, suppose C has empty interior. Since it is closed and convex, it should be contained in an affine space of lower dimension. Then the normal to that affine space will be contained in C^*. Actually, any scalar multiple of normal will be contained in C^*. Hence, C^* contains a line.

From the previous two paragraphs the cone C has non-empty interior iff C^* does not contain a line. Now, suppose C^* is empty. It implies $C^{**} = C$ contains a line. The contrapositive is that if C does not contain a line then C^* is non-empty.

Hence, it is clear that the dual of a proper cone is proper.

4 Dual cone of a finitely generated cone

Another interesting case of a dual cone is the case of a finitely generated cone. Suppose we are given a cone $C = cone(x_1, \cdots, x_k)$. How will its dual cone look?

$$C^* = \{y : y^T x \geq 0, \ \forall x \in C\}$$

Clearly, for all the generators of C $(x_1, \cdots, x_k)$, $y^T x_i \geq 0$. Since any element of the cone is a conic combination of these vectors, this condition is actually sufficient too. So,

$$C^* = \{y : y^T x_i \geq 0, \ i = 1, \cdots, k\}$$

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Notice that this gives the definition of cone in terms of inequalities. Affine Weyl theorem tells us that these two are equivalent formulations. So C^* is also a finitely generated cone. Using the proof of Affine Weyl theorem, we can construct the inequalities which will specify C^*.

It is important to note that the generators of the original cone are the inequalities for the dual cone and vice versa. So in the case of finitely generated cones, it is easy to specify the dual cone.

Exercise 5. What will be the dual cone in $\mathbb{R}^2$ generated by two vectors.