Lecture 5: Properties of convex sets

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This week we will see many other properties of convex sets. These properties make convex sets special and are the reason why convex optimization problems can be solved much more easily as compared to other general optimization problems.

1 Intersections and unions of convex sets

Suppose we are given two convex sets S_1 and S_2. What happens when we take their intersection or union. Intersection of two convex sets is convex. Consider two points in the intersection $S_1 \cap S_2$. They are contained in the individual sets S_1, S_2. So the line segment connecting them is contained in both the sets S_1, S_2 and hence in the set $S_1 \cap S_2$.

The same argument can be extended to intersection of finite number of sets and even infinite number of sets too. Since a polytope is an intersection of halfspaces and hyperplanes (linear inequalities and linear equalities), it gives an easier proof that a polytope is convex.

But the same property does not hold true for unions. In general, union of two convex sets is not convex. To obtain convex sets from union, we can take convex hull of the union.

Exercise 1. Draw two convex sets, s.t., there union is not convex. Draw the convex hull of the union.

2 Minkowski sum

We can define another operation on sets to form a new set. A Minkowski sum of two sets S_1, S_2 is the set formed by taking all possible sums such that first vector is from S_1 and second vector is from S_2.

$$S_1 + S_2 = \{x : x = x_1 + x_2, x_1 \in S_1, x_2 \in S_2\}$$

It can be defined more generally for a finite family of sets too. In general, Minkowski sum of two convex sets is convex (prove it). Actually it behaves really nicely with respect to taking convex hull operation.

$$Conv(S_1 + S_2) = Conv(S_1) + Conv(S_2)$$

Exercise 2. Is this property satisfied by intersection?

3 Projection

Exercise 3. Suppose a set $S \subseteq \mathbb{R}^m\mathbb{R}^n$ is convex. Prove that $T = \{x_1 \in \mathbb{R}^m : (x_1, x_2) \in S\}$ is convex.

4 Product

Exercise 4. Suppose sets $S_1 \subseteq \mathbb{R}^m,\ S_2 \subseteq \mathbb{R}^n$ are convex. Prove that $S_1 \times S_2 = \{(x_1, x_2) : x_1 \in S_1,\ x_2 \in S_2\}$ is convex.

* Thanks to books from Boyd and Vandenberghe, Dantzig and Thapa, Papadimitriou and Steiglitz

5 Preservation under affine functions

Lets define a class of functions called *affine functions*. A function $f : \mathbb{R}^n \to \mathbb{R}^k$ is called affine iff $f = Ax + b$ for some $A \in \mathbb{R}^{m\times n},\ b \in \mathbb{R}^m$.

This affine functions act nicely on convex sets. It is easy to show that the image of a convex set under affine functions is convex. Given S is convex, set $T = \{x : Ay + b = x,\ y \in S\}$ is also convex. It turns out that even the pre-image under an affine function is convex.

Theorem 1. *Given a set S to be convex, the set*

$$T = \{x : Ax + b = y,\ y \in S\}$$

is also convex.

Proof. Suppose there are two elements t_1, t_2 of T. Then there exist s_1, s_s, s.t., $At_1 + b = s_1, At_2 + b_2 = s_2$. Taking the convex combination of these two equalities,

$$A(\theta t_1 + (1-\theta)t_2) + b = \theta s_1 + (1-\theta)s_2.$$

Hence $\theta t_1 + (1-\theta)t_2$ is in T for all $0 \le \theta \le 1$. □

6 Generalized inequalities

We learned about cones as an example of convex sets. Most of the cones encountered in this course will be a special subclass called proper cones. A cone C is proper iff

- C is convex.
- C is closed (it has boundaries). Remember that a cone is not bounded.
- C has nonempty interior.
- C is pointed, i.e., it does not contain any line.

In case of real numbers, one is interested in $\geq, \leq$ kind of ordering. Using a proper cone, we can define an ordering (partial) over the elements of $\mathbb{R}^n$. A partial ordering on a cone C can be defined as

$$x \succeq_C y \Leftrightarrow x - y \in C$$

Exercise 5. What is the cone for our usual ordering of real numbers $\leq, \geq$?

Why did we define these generalized inequalities in terms of cones and not say polytope. Why did we take cone to be of special form. The reason is that we need this new ordering to be similar to old familiar ordering over real numbers. This new ordering is going to satisfy many properties we are familiar with (though they are not going to satisfy all properties, for one this ordering is partial).

- If $x \succeq_C y$ and $u \succeq_C v$, then $x + u \succeq_C y + v$.
- Transitivity: if $x \succeq_C y$ and $y \succeq_C z$ then $x \succeq_C z$.
- If $x \succeq_C y$ and $\theta \geq 0$, then $\theta x \succeq_C \theta y$.
- $x \succeq_C x$.
- If $x \succeq_C y$ and $y \succeq_C x$, then $y = x$.
- If $x_i \succeq_C y_i$ as $i \to \infty$ then $x \succeq_C y$, where x, y are respective limits.

We can define strict ordering $\succ$ using the same idea, $x \succ_C y$ iff $x - y \in Int(C)$. Here $Int(C)$ is the interior of the cone (Cone - points on the boundary of Cone). This strict ordering also has many similar useful properties.

- If $x \succ_C y$, then $x \succeq_c y$

– If $x \succ_C y$ and $u \succeq_C v$, then $x + u \succ_C y + v$.
– If $x \succ_C y$ and $\theta \geq 0$, then $\theta x \succ_C \theta y$.
– Important: If $x \succ_C y$, then for small enough u, v, $x + u \succ_C y + v$.

Exercise 6. For all these properties, prove them and see what properties of cones did we use.

The idea is to allow these *special inequalities* in our optimization constraints. An optimization problem over a cone C is a problem where the constraints are either equalities/ inequalities or generalized inequalities over that cone. Intuitively, the properties of proper cones and convex sets allow the efficient solution of these optimization problem. Semidefinite programming is optimization over cone called semidefinite cone.

Most of the material on generalized inequalities has been directly taken from Boyd and Vandenberghe book. Detailed description of this topic can be found in the book of Boyd and Vandenberghe.

7 Characterization of polytope

We saw last week that cones and bounded polytope has two representations. One in terms of linear inequalities and one in terms of convex combinations (conic combination). There is a similar theorem for polytope.

Theorem 2. *Let there be a polytope defined by a set of inequalities, $P = \{x \in \mathbb{R}^n : Ax \leq b\}$. Then there exist vectors $x_1, \cdots, x_k \in \mathbb{R}^n$ and $y_1, \cdots, y_l \in \mathbb{R}^n$, s.t.,*

$$P = Cone(x_1, \cdots, x_k) + Conv(y_1, \cdots, y_l)$$

This is known as the *Affine Minkowski-Weyl* theorem. We will not do the proof of this theorem in this course. Notice that now there are equivalent characterizations of cone/polytope/bounded polytope in terms of convex/conic hulls or linear inequalities. It is instructive to remember the special forms of linear inequalities and hulls required to make these shapes.