Lecture 4: Examples of Convex sets

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We last time looked at the definition of convex sets. Today we look at some of the examples which will be really useful in the future.

1 Polytopes and Polygons

We will mostly be interested in linear equations and linear inequalities as our constraints. We already discussed the set of points which satisfy the set of all constraints is called the *feasible region*. When this constraints are linear inequalities and linear equalities the feasible region is called a polytope. Mathematically, S is a polytope iff

$$S = \{x : a_i^T x - b_i \leq 0 \ \ i = 1, 2, \cdots, m \text{ and } c_i^T x - d_i = 0 \ \ i = 1, 2, \cdots, p\}$$

A bounded polytope in two dimensions is called a polygon.

Exercise 1. Prove that the polytope is a convex set.

Exercise 2. What are the polytopes in one dimension?

Geometrically, polytopes are intersections of hyperplanes and halfspaces. Remember that a polytope could be bounded or unbounded. Intuitively, polytopes have vertices, edges, planes and hyperplanes as their bounding surface. A bounded polytope can be thought of as the convex hull of its vertices. Actually a bounded polytope can have an alternate definition as the convex hull of a finite set of points. The statement that these two definitions (the previous one and the original definition in terms of equalities and inequalities) are the same is known as Minkowski-Weyl theorem. The proof of that is out of the scope of this course. Now we look at a special case of polytope.

We haven't defined vertices/extremal points formally till now. It is intuitively clear that a vertex is a corner of the polytope. Formally, A vertex of a polytope is the point which cannot be expressed as the convex combination of two different points in the polytope. This implies that vertex is not *inside* of any line segment joining two points in the convex set.

We saw that the convex hull of a triangle and a point inside the triangle is triangle itself. Suppose we are given a convex set and we want to find out the minimal set whose convex combinations will generate the entire set. By the definition of vertices, all vertices should be in this minimal set. It turns out that the for a bounded polytope this set (set of all vertices) is enough.

Exercise 3. Suppose $S = Conv(x_1, \cdots, x_k)$. Prove that x_i is not extremal/vertex if and only if it can be written as the convex combination of other x_j's.

2 Simplex

Given $k+1$ points $x_0, x_1, \cdots, x_k \in \mathbb{R}^n$, s.t., $\{x_1 - x_0, x_2 - x_0, \cdots, x_k - x_0\}$ are linearly independent. Then the convex hull of these $k+1$ points $Conv(x_0, x_1, \cdots, x_k)$ is called a simplex. It is a generalization of the idea of a triangle or tetrahedron.

Theorem 1. *A simplex is a polyhedron.*

* Thanks to books from Boyd and Vandenberghe, Dantzig and Thapa, Papadimitriou and Steiglitz

Proof. Sketch: We construct an $n \times k$ matrix $M = (x_1 - x_0, x_2 - x_0, \cdots, x_k - x_0)$. Any element in the simplex can be represented as

$$x = x_0 + M\theta \tag{1}$$

Here $\theta \in \mathbb{R}^k$ is positive and inner product of θ with all 1's vectors is less than 1. Now M being full rank implies that there exist matrix N which can diagonalize M. We can multiply Eq. 1 by matrix N and get the linear inequalities and equalities by removing θ.

Exercise 4. Fill in the details of the proof.

□

3 Cones

We have seen sets made by vertices, lines, line segments. Now we look at sets generated by *rays*. A set is called a cone iff every ray from origin to any element of the set is contained in the set. Hence the set C is a cone iff for every $x \in C$ we have $\theta x \in C, \theta \geq 0$.

Notice that a cone is *not* a set which has all possible conic combinations of all its points. Remember the notion of conic combination. A conic combination of vectors $x_1, \cdots, x_k \in \mathbb{R}^n$ is any vector of the form $\theta_1 x_1 + \cdots + \theta_k x_k$ for $\theta_1, \cdots, \theta_k \geq 0$.

The previous paragraph implies that a cone is not necessarily convex (Give example of a cone which is not convex). A set which is a cone and is convex is called a convex cone. In this course we will mostly be concerned with convex cones. Mathematically, a convex cone C is a set where for all $\forall x_1, x_2 \in C$ and $\theta_1, \theta_2 \geq 0,\ \theta_1 x_1 + \theta_2 x_2 \in C$. So we get the definition, A cone is convex iff it contains all the conic combinations of its elements.

Convex hulls and cones are closely related.

Exercise 5. Take x_i's as row vectors. Prove,

$$x \in Conv(x_1, x_2, \cdots, x_k) \Leftrightarrow (x, 1) \in Cone((x_1, 1), \cdots, (x_k, 1))$$

Note: Some authors define cones as sets closed under *positive* scalar multiplication. We have defined cones as sets closed under non-negative scalar multiplication.

Clearly the set $Cone(x_1, x_2, \cdots, x_k) := \{\theta_1 x_1 + \cdots + \theta_k x_k :\ \forall i\ \theta_i \geq 0,\ x_i \in \mathbb{R}^n\}$ is a cone. It is called a finitely generated cone because it is generated by finite number of vectors. A convex finitely generated cone is also a polytope. Next theorem gives a characterization of a finitely generated cone.

Theorem 2. *Weyl's Theorem: A non-empty finitely generated convex cone is a polytope.*

Proof. Suppose the set of generators for cone C are $x_1, \cdots, x_k$. We can define a matrix X which has x_i's as the columns. Then the cone C can be written as

$$C = \{x : x = X\theta,\ \theta \in \mathbb{R}_+^k\}$$

Converting equalities into inequalities

$$C = \{x : x - X\theta \leq 0,\ X\theta - x \leq 0,\ \ -\theta \leq 0$$

Now θ can be eliminated from these inequalities using something knows as Fourier-Motzkin elimination.

Lemma 1. *Let $Ax \leq b$ be a system of m inequalities in n variables. This system can be converted into another equivalent system $A'x \leq b'$ with $n-1$ variables and polynomial in m many inequalities. Here equivalent means any solution x of old system will be a solution of the new system ignoring the removed variable. Also given any solution x of new system ($A'x \leq b'$), we can find a solution (x_0, x) of old system.*

Proof. Suppose the removed variable is x_0. We divide all the inequalities into three sets depending upon whether the coefficient of x_0 is positive (P), negative (N) or zero (Z). Divide the inequalities in P and N by the modulus of the coefficient of x_0. The inequalities in the new system are the inequalities from Z and every inequality of the form $p_i + n_j \leq 0, p_i \in P,\ n_j \in N$.

Exercise 6. Prove that this construction works.

□

With the θ eliminated from the system of equations which define the cone, we get

$$C = \{x : Ax \leq 0\}$$

Hence it is a polytope. □

Note: A general polytope is $Ax \leq b$ and we will see that a finitely generated cone is $Ax \leq 0$. We defined these sets polytopes, cones, simplex etc.. They are interesting because the feasible regions of our optimization problems will be intersections of these various kind of convex sets.