Lecture 3: Convex sets

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We denote the set of real numbers as $\mathbb{R}$. Most of the time we will be working with space $\mathbb{R}^n$ and its elements will be called vectors. Remember that a subspace is a set of vectors closed under addition and scalar multiplication.

First we learn how to take interesting combinations of a given set of vectors. For vectors $x_1, x_2, \cdots, x_k$; any point y is a linear combination of them iff

$$y = \alpha_1 x_1 + \alpha_2 x_2 \cdots + \alpha_k x_k \ \ \forall i, \ \alpha_i \in \mathbb{R}$$

If we restrict α_i's to be positive then we get something called a conic combination.

$$y = \alpha_1 x_1 + \alpha_2 x_2 \cdots + \alpha_k x_k \ \ \forall i, \ \alpha_i \geq 0 \in \mathbb{R}$$

Instead of being positive, if we put the restriction that α_i's sum up to 1, it is called an affine combination

$$y = \alpha_1 x_1 + \alpha_2 x_2 \cdots + \alpha_k x_k \ \ \forall i, \ \alpha_i \in \mathbb{R}, \ \sum_i \alpha_i = 1$$

When a combination is affine as well as conic, it is called a convex combination.

$$y = \alpha_1 x_1 + \alpha_2 x_2 \cdots + \alpha_k x_k \ \ \forall i, \ \alpha_i \geq 0 \in \mathbb{R}, \ \sum_i \alpha_i = 1$$

Exercise 1. What is the linear/conic/affine/convex combination of two points in $\mathbb{R}^2$?

1 Affine sets

Lets start by defining an affine set.

Definition 1. *A set is called "affine" iff for any two points in the set, the line through them is contained in the set. In other words, for any two points in the set, their affine combinations are in the set itself.*

Theorem 1. *A set is affine iff any affine combination of points in the set is in the set itself.*

Proof. Exercise. (Use induction) □

Exercise 2. What is the affine combination of three points?

Suppose the three points are x_1, x_2, x_3. Then any affine combination can be written as $\theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3, \ \sum_i \theta_i = 1$. We can write the expression as $\theta_1(x_1 - x_3) + \theta_2(x_2 - x_3) + x_3$. Since θ_1, θ_2 are unconstrained now, this sum can be thought of intuitively as (x_3 + plane generated by $x_1 - x_3$ and $x_2 - x_3$). This gives a nice relation between affine and linear combinations.

1.1 Examples of affine sets

- Offset + Subspace: Any affine set can thought of as an offset x added to some subspace $S \sim (V = \{v + x :\sim v \in S\})$. It is easy to see that such a set is affine. Also given an affine set V and a point inside it x; it can be shown that $S = \{v - x : \ v \in V\}$ is a subspace. Notice that the subspace associated with an affine set does not depend upon the point chosen (exercise). So we can define the dimension of the affine set as the dimension of the subspace. So it turns out that *offset+subspace* is not just an example, but a characterization of affine sets.
- *Exercise 3.* Show that the feasible region of a set of linear equations is affine.

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* Thanks to books from Boyd and Vandenberghe, Dantzig and Thapa, Papadimitriou and Steiglitz

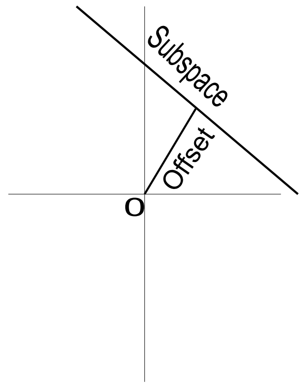

Fig. 1. Example of an affine set

2 Convex set

From the definition of affine sets, we can similarly guess the definition of convex sets.

Definition 2. *A set is called convex iff any convex combination of a subset is also contained in the set itself.*

Theorem 2. *A set is convex iff for any two points in the set their convex combination (line segment) is contained in the set.*

We can prove this using induction. It is left as an exercise.

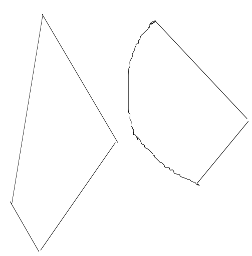

Fig. 2. Example of convex sets

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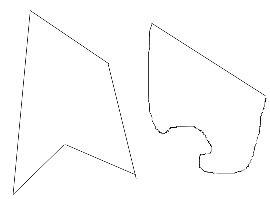

Fig. 3. Example of non-convex sets. How can we make them convex?

2.1 Convex hull

Convex hull of a set of points C (denoted $Conv(C)$) is all possible convex combinations of the subsets of C. It is clear that the convex hull is a convex set.

Theorem 3. *$Conv(C)$ is the smallest convex set containing C.*

Proof. Suppose there is a smaller convex set S. Then S contains C and hence all possible convex combinations of C. So S contains $Conv(C)$. But then S is not bigger than $Conv(C)$. This implies $S = Conv(C)$. $\square$

Convex hull of S can also be thought of as the intersection of all convex sets containing S (Prove it).

Exercise 4. What is the convex hull of three vertices of a triangle? What if I add a point inside the triangle? What about outside? What can we infer about the properties of convex hull from these examples?

3 Lines, Hyperplanes and Halfspaces

Probably the simplest examples of convex set are $\varnothing$(empty set), a single point and $\mathbb{R}^m$(the entire space). The first example of a non-trivial convex set is probably a line in the space $\mathbb{R}^n$. It is all points y of the form

$$y = \theta x_1 + (1 - \theta)x_2$$

Where $x_1$and x_2 are two points in the space and $\theta \in \mathbb{R}$ is a scalar. This gives us the unique line passing through x_1 and x_2. If we constraint θ to be in $[0, 1]$, it is called the line segment between x_1 and x_2.

So, a point is on the line segment between x_1 and x_2 iff it is a convex combination of the given two points. Note that condition for being a convex set is weaker than the condition for being an affine set. Hence an affine set is always convex too. Since line is an affine set, it is a convex set too. A line segment will be convex set but not affine.

We extend the idea of a line to hyperplanes and halfspaces. A hyperplane is described by a vector $a \in \mathbb{R}^n$ and a number $b \in \mathbb{R}$

$$\{x : a^T x - b = 0\}$$

Exercise 5. Prove that a hyperplane is affine and so convex too.

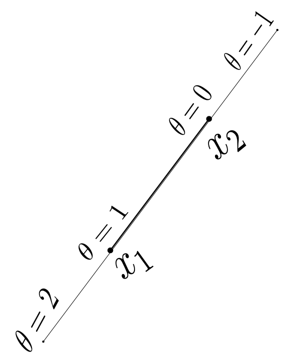

Fig. 4. A line, line segment, points on the line and θ's corresponding to them

Suppose we know a point x_0 on the hyperplane H. Then this equation can be changed to $a^T x = b \Rightarrow a^T x = a^T x_0,\ x_0 \in H$. We can view it as the set of all points which have a constant b inner product with a. This gives a very nice geometrical picture of the hyperplane, i.e., all points in H can be expressed as the sum of x_0 and a vector orthogonal to a (we can call it $a^{\perp}$). So, another definition of hyperplane is

$$\{x : x = x_0 + a^{\perp}, a^T a^{\perp} = 0, x_0 \in H\}$$

Note that this definition assumes that we know a point on the hyperplane. But this point is not special in any way, for any point on the hyperplane we can define the hyperplane in the same way. The vector a is called the *normal* vector of the hyperplane and b is called the *offset*. Another way to think of this hyperplane is to take the set of all vectors orthogonal to a (Hyperplane, passing through origin) and offset them by distance b.

We studied the generalization of a line in higher dimensional space, it was called a hyperplane. A hyperplane divides the space into two parts, $a^T x \geq b$ and $a^T x \leq b$. Geometrically, they are the two sides of the plane.

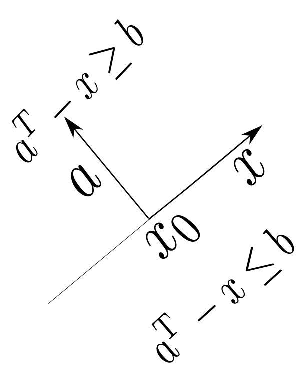

Fig. 5. Halfspace with normal vector a and offset b.

Exercise 6. Is the halfspace affine? convex?